HANDBOOK OF OPTICS

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FOR

STUDENTS OF OPHTHALMOLOGY

BY

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PREFACE

In the following work there is presented so much of the science of optics as pertains directly to ophthalmology. Simplicity has been sought so far as this is not incompatible with thoroughness; for whoever would become versed in ophthalmology as a *science* must in the beginning make the mental effort necessary to acquire a clear understanding of the refraction of light through a compound optical system such as the eye.

The demonstrations, some of which may appear formidable to the student, require no knowledge of mathematics beyond that of simple algebraic equations and the elementary truths of geometry. For those who may not be familiar with the trigonometrical ratios, a brief synopsis has been furnished in an appendix.

In demonstrating refraction by prisms and by spherical surfaces, Heath's "Geometrical Optics" has been used as a basis, but many modifications have been made.

A uniform notation, with which the student will easily become familiar, has been preserved throughout the book; by this means those who may be indisposed to follow the algebraic processes in detail will be aided in understanding the methods of demonstration.

WASHINGTON, June, 1899.

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HANDBOOK OF OPTICS

INTRODUCTION

That branch of physical science which treats of light and vision is called Optics. It may be subdivided into **Geometrical**, **Physical**, and **Physiological** optics. Geometrical optics deals with the theory of light; it is a "mathematical development" of the experimental laws by which light is supposed to be controlled,—the laws of reflexion and refraction, and the supposition that light travels through homogeneous media in straight lines. Physical optics investigates the causes and nature of light; while Physiological optics treats of the phenomena of vision or the sensation produced by the action of light falling upon the retina.

Catoptrics and **Dioptrics**, terms less used now than formerly, refer respectively to the phenomena of reflexion and refraction of light. The science of optics was practically unknown to the ancients. They had observed that light travels in a homogeneous medium in straight lines, and they also knew the simple law of reflexion and the focusing property of

lenses and mirrors; but their ideas of vision were most crude, it being commonly supposed that light was something given out from the eye. Strange to say, this theory did not entirely disappear for many centuries.

Spectacles of spherical lenses were probably introduced in the thirteenth century. To a spectacle maker, Hans Lippershey, is ascribed the first telescope in 1608, and in the following year Galileo independently constructed his telescope; but with the astronomer Kepler, who died in 1630, begins the true science of optics. Willebrod Snellius, professor of mathematics in Leyden, discovered the law of refraction; and to Sir Isaac Newton is due the discovery that white light is composed of various colors, capable of separation by the action of a prism.

Up to this time the propagation of light was thought to be instantaneous. Römer, a Danish astronomer, discovered in 1676 that time is required for the transmission of light. This he inferred from discrepancies between the calculated and actual time in the observation of eclipses of Jupiter's satellites; and he rightly attributed these discrepancies to the unequal distances through which light had to travel, owing to the varying distance between the planet and the earth. Terrestrial measurement of the rate of propagation of

light was not accomplished until the middle of the present century. This was done with instruments of great precision by Fizeau and Foucault, and more recently by Professor Newcomb. The velocity as thus determined is 300,000,000 metres or 186,000 miles per second.

The question as to the method of transmission of light has been the subject of much controversy. Leaving aside the speculations of the ancients, the two theories are the Corpuscular or Emission Theory and the Wave Theory. The first supposes a luminous body to give off certain particles, which, striking the eye, produce vision. The wave theory supposes that all space is pervaded by a substance called **ether**; and that by means of this substance, waves, excited in the luminous body, are transmitted to the eye. While some of the phenomena of light can be explained by either theory, experiments by Huyghens, Young, and Fresnel have rendered the emission theory untenable. While we are forced to the belief that light is propagated in waves, we are ignorant as to the nature of these waves. It was formerly supposed that the ether was a highly elastic body, transmitting vibrations of its particles through space just as a rod of steel, if struck near one end, will convey the vibrations to the other end.

To explain the phenomena of light it is necessary to suppose the vibrations transverse to the direction

of propagation, as is the case in the illustration cited. The waves of sound travelling through air differ from these in that in the case of the sound waves the particles of air vibrate to and fro in the direction of propagation.

The modern study of electricity has changed our conception of ether waves; for according to the electro-magnetic theory of waves, each particle of ether is "polarized" or charged with energy, which in turn is transmitted to the next particle, and so on. These waves of energy produce various effects depending upon the rapidity of vibration. Those of least rapidity are manifested as electricity, and next in order come heat-producing waves. As the rapidity increases we have light of different colors, red being the color of least rapidity of vibration. Below this color in the spectrum are found heat waves. Violet is the color of greatest rapidity, and beyond this no light will be seen, but certain chemical effects produced there indicate the presence of waves capable of causing chemical action. It is also thought probable that gravitation is exercised by the transmission of energy through ether waves. For the further study of the wave theory, which is necessary for the proper understanding of interference phenomena and of polarized light, the student is referred to complete treatises on the theory of light.

The investigations in the following chapters are based upon the experimental laws of reflexion and refraction, and upon the supposition that light travels in a homogeneous medium in straight lines.

CHAPTER I

REFRACTION AT PLANE SURFACES

A luminous body emits light in all directions. That portion of light which travels along a particular line is called a **ray**. A collection of rays which do not deviate far from a central fixed ray is called a **pencil**. When a ray of light passing through a medium meets another medium of different density, it is divided into two portions; a part of the light is reflected back into the first medium, while the remaining portion passes into the second medium, and is generally altered in direction. Of the reflected light, a part is said to be *regularly* reflected and a part *scattered*. Strictly speaking, all reflexion is regular; but owing to the unevenness of the surface, the light is reflected in various directions. It is by this means that we see a non-luminous object; for if the light were all regularly reflected, we should see only the image of the illuminating source. The more even the surface, the greater is the regularly reflected and the less the scattered light. If the substance is opaque, no light passes into it; the incident light is either

reflected or absorbed. When a ray passes from one medium to another, the two portions of the ray before and after entering the new medium are called, respectively, the **incident** and the **refracted** ray; and the acute angles which they make with the normal to the surface are called, respectively, the **angle of incidence** and the **angle of refraction.** In

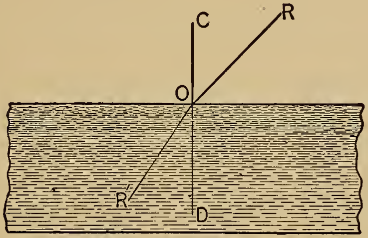

FIG. 1.

Fig. 1 RO is the incident ray, OR' is the refracted ray, COR is the angle of incidence, and DOR' is the angle of refraction. It is found by experiment that the incident and refracted rays always lie on opposite sides of the normal to the refracting surface; that the angles of incidence and refraction always lie in the same plane; and that the sine of the angle of incidence always bears a fixed ratio to the sine of the angle of refraction. This ratio, while fixed for the same two media, varies with the nature of the refracting substances. It is called the **refrac-**

tive index for the two media. This law of refraction is called **Snell's** law; it is sometimes called **Descartes'** law, since he first published it in its present form. For many years prior to Snell's discovery, investigators had constructed tables expressing the relation between angles of incidence and refraction, but even Kepler was unable to deduce from these the law governing this relation. This

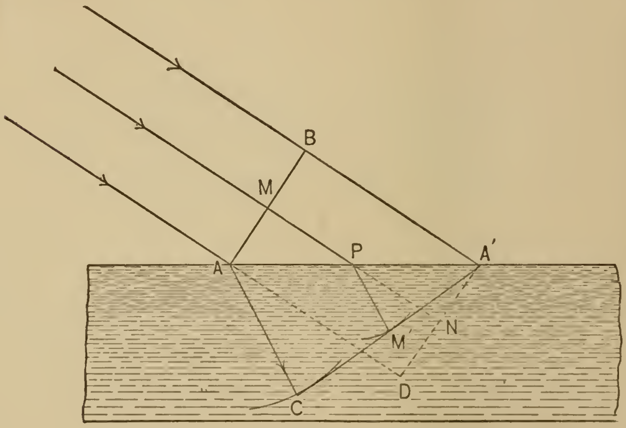

Fig. 2.

law, discovered by experiment, acquires new interest from its corroboration of the wave theory of light. It was supposed — and modern experiments have proved it to be true — that light travels with different velocities in media of different densities. Let

AB, Fig. 2, be a small portion of the front of a wave of light which proceeds from a distant point; then AB, being an indefinitely small arc of a circle, is indistinguishable from a straight line, and may be regarded as such. AA' represents the surface of refraction; v the velocity of light in the first medium; v' the velocity in the second medium. Then if t is the time required by the light to traverse the distance BA', we have $BA' = v \cdot t$, which is also equal to AD. If the wave had been unobstructed by the second medium, it would occupy the position $A'ND$ at the end of the time t. But the portion of the wave front at A, meeting the more dense refracting substance, does not travel so fast as the portion BA'. The point A becomes the centre of a wave disturbance, which in the time t has reached the point C; and, consequently, $AC = v' \cdot t$. Similarly, the portion of the wave at P travels in the second medium the distance PM', while it would have travelled in the first medium the distance PN.

Therefore, $AC : AD = v' : v$, and $PM' : PN = v' : v$; from which

$$\frac{AC}{AD} = \frac{v'}{v} = \frac{PM'}{PN}; \text{ or, } \frac{AC}{PM'} = \frac{AD}{PN} = \frac{AA'}{PA'}.$$

From this equation it follows that $PA'M'$ and $AA'C$ are similar triangles, and consequently M lies on the line $A'C$. Since in like manner any

other point of the wave front will at the end of the time t lie on the line $A'C$, which is perpendicular to AC, then $A'C$ will represent the wave front after refraction, and AC will represent the direction in which the refracted light travels.

If i be the angle of incidence of the ray, then $BA'A = 90 - i$, and $AA'D = i$. If r be the angle of refraction, then $A'AC = 90 - r$, and $AA'C = r$.

From the triangle $A'AD$ we have

$$\sin i = \frac{AD}{AA'},$$

and from $AA'C$ we have

$$\sin r = \frac{AC}{AA'};$$

from which $$\frac{\sin i}{\sin r} = \frac{AD}{AC} = \frac{v}{v'} = n.$$

From this we see that the constant ratio between the sines of the angles of incidence and refraction is that of the velocity of light in the first medium to the velocity in the second medium. We learn from experiment that the deviation is toward the normal to the surface when the ray passes from a rarer to a denser medium, and away from the normal when the ray passes from a denser to a rarer medium. If the deviation is toward the normal, then i is greater than r, and consequently v is greater than v'. Thus,

according to the wave theory, the velocity of light must be greater in air than in a dense medium, such as water or glass. On the other hand, according to the emission theory, the velocity must be greater in a dense medium than in a rare one.* But experiments have proved that the velocity is less in dense than in rare media; and the evidence is in favor of the wave theory.

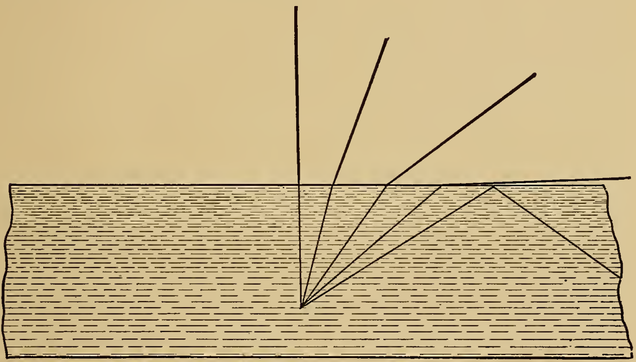

FIG. 3.

When light passes from a rarer to a denser medium, n is greater than unity, and since $\sin r = \frac{\sin i}{n}$, $\sin r$ is never greater than unity whatever may be the angle i; but when light passes from a denser to a rarer medium, n is less than unity, and, for certain values of i, $\sin r$ may become greater than unity. As

* Preston's "Theory of Light," 2d ed., p. 17.

the sine of an angle cannot be greater than unity, this would be an impossible value for r. It is found by experiment that when i has such value as to make $\sin r$ greater than unity, light does not pass out of the denser medium, but is reflected back into this medium. This is called the **total internal reflexion**, and the angle of incidence which makes $\sin r$ equal to unity is called the **critical** angle. A glance at Fig. 3 will show the meaning of this. It is also found, as we might expect from this phenomenon, that as the angle of incidence increases the proportion of reflected light increases, while that of refracted light diminishes. Advantage is taken of the phenomenon of total internal reflexion in the construction of certain optical instruments.

Another experimental fact is the reversibility of the path of light, that is, if the direction of a ray be reversed so that the angle of refraction becomes the angle of incidence, then the original angle of incidence will become the new angle of refraction. This being so, it is evident that a ray after refraction through a medium with parallel surfaces will, upon reëntering the original medium, be parallel to its direction before refraction. It will, however, undergo a lateral displacement varying with the thickness of the medium. See Fig. 4.

We have learned that $\frac{\sin i}{\sin r} = n$, where n is the

ratio of the velocity of light in the first medium to its velocity in the second medium. If the velocity in a vacuum is taken as the standard, the ratio of the velocity in any medium to the velocity in a vacuum is called the **absolute** refractive index. The ratio of the velocities for any two media may therefore be expressed in terms of the absolute indices of the two media. Thus, if n be the absolute index

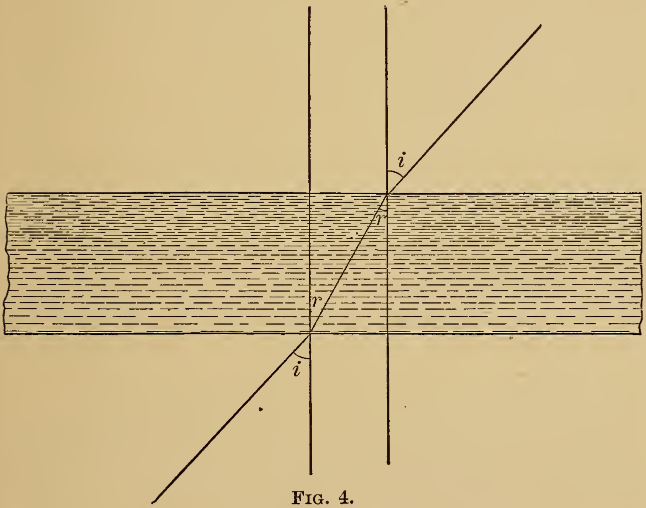

FIG. 4.

of the first medium and n' that of the second, the relative index for the two media will be $\frac{n'}{n}$. Snell's law thus becomes $n \cdot \sin i = n' \cdot \sin r$, and it is usually written in this form.

In any refraction the greater the angle of incidence, the greater will be the deviation; and the greater the angle of incidence, the greater will be the increase in deviation for a fixed increase in the angle of incidence. This follows directly from the equation $\frac{\sin i}{\sin r} = n$. Reference to Appendix I. will render this clear, for it is there shown that the sine of an angle increases less rapidly than the angle; and the greater the angle, the less will be the change effected in its sine by a fixed increase of the angle. Hence, if r be smaller than i, a smaller increase in r will be required to maintain the constant ratio between $\sin i$ and $\sin r$ than in the greater angle i; and this is true in a greater degree as i approaches 90 degrees. Since the deviation is expressed by $i - r$, it follows that this increases when i increases. As the path of light is reversible, the same holds true when i is less than r, that is, when the ray passes from a denser to a rarer medium.

A medium bounded by two plane faces meeting in an edge is called a **prism**. At present we shall consider only the refraction of *rays* through prisms, reserving for a future chapter the more difficult subject of refraction of *pencils* of light. We shall suppose the rays to lie in a **principal plane** of the prism, that is, in a plane which is perpendicular to the edge of the prism, and consequently to the plane of each

face of the prism. We shall also confine our attention to prisms whose refractive index is greater than that of air. When a ray of light passes through such a prism, the deviation is in all cases from the apex toward the thicker part of the prism. We have seen that a ray, passing through a plane

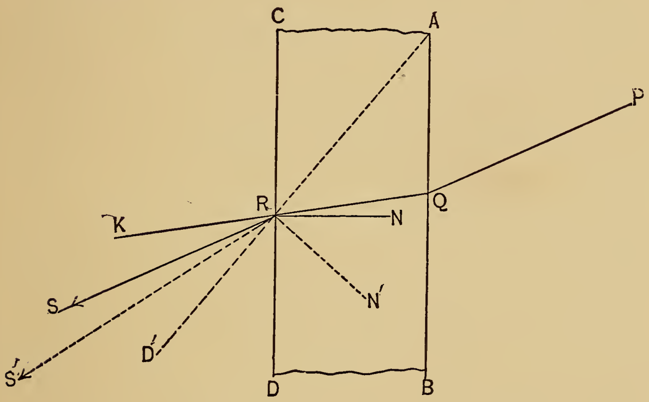

FIG. 5.

bounded by parallel surfaces, emerges without deviation. Let *PQRS* (Fig. 5) represent a ray passing through the medium *ABCD*, *AB* and *CD* being parallel. *RN* is normal to the face *CD*. Now suppose the face *CD* be turned into the position *AD'*, making the prism *BAD'*; then the normal *RN* must turn into the position *RN'*; and by this change the angle of incidence is increased from *NRQ* to *N'RQ*.

We have learned that with an increase of the angle of incidence there is also an increase of deviation. *KRS* represents the deviation at the second surface when the face has the position *CD*, and as the deviation is increased by turning the face into the position *AD'*, then *KRS'*, greater than *KRS*, will represent the deviation at the second face of the prism *BAD'*. *RS* is parallel to *PQ*, the direction of the ray be-

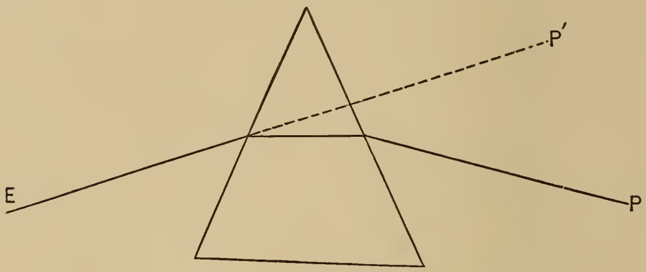

FIG. 6.

fore entering the prism; hence *SRS'* represents the deviation of the ray in its passage through the prism, and this deviation is away from the apex of the prism.

Light from an object at *P* (Fig. 6) would be so deviated as to enter an observer's eye at *E*, and the object would appear to be at *P'*. Hence an object seen through a prism is displaced toward the apex of the prism.

All light is not equally deviated by prisms. If a

narrow beam of sunlight be passed through a prism in a darkened room, and the refracted light be intercepted by a screen, it will be found that the beam has been decomposed into bands of colored light. These are violet, indigo, blue, green, yellow, orange, and red. Of these, violet is most and red least deviated. They are called the **colors of the spectrum.** Since the deviation of a ray by refraction is explained by the theory that the passage of light is retarded upon entrance into a substance of greater density, it is necessary to suppose that this retarding power is different for different colors; violet, which is most deviated by the prism, must suffer the greatest retardation, and red, the color of least deviation, must be least retarded by the prism. It is supposed that only dense substances offer this unequal resistance to the passage of light of different colors, and that in space and in air all light travels with the same velocity.*

The property which prisms possess of separating colors is called **dispersion.** It is a most important property, but only incidentally concerns the student of ophthalmology. The chief use of prisms in ophthalmological practice is for the purpose of changing the apparent position of objects. In the weaker prisms dispersion is not noticeable; but if an opaque

* Preston's "Theory of Light," 2d ed., p. 97.

object be viewed through a prism of considerable deviating power, it will be tinged with red toward the apex and with violet toward the base of the prism.

Let $PQRS$ (Fig. 7) represent a ray passing through the prism whose apex is at O, and whose faces are inclined at an angle a. This angle is called the **refracting angle** of the prism. At Q and

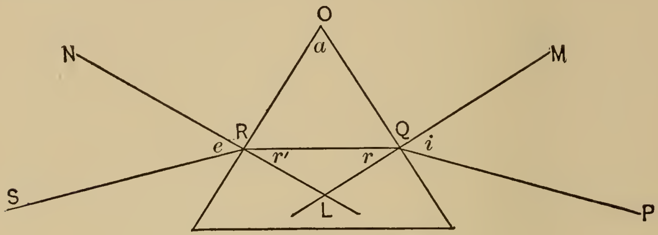

FIG. 7.

R draw the normals LQM and LRN. Let the angle of incidence PQM be called i, and the angle of refraction LQR be called r. Also let LRQ be r', and NRS be e; then from the law of refraction we have $\sin i = n \cdot \sin r$, and $\sin e = n \cdot \sin r'$. The angle ORQ is equal to $90 - r'$ and OQR is equal to $90 - r$. Since the sum of the three angles of the triangle ROQ must be equal to 180 degrees, we have

$$a + 90 - r + 90 - r' = 180 \text{ ; or, } r + r' = a.$$

The deviation of the ray at the first surface is

represented by $i - r$, and at the second surface by $e - r'$. The total deviation is denoted by $i + e - (r + r')$, or $i + e - a$.

Hence the deviation produced by a prism is equal to the sum of the angles of incidence and emergence, minus the refracting angle of the prism.

Let us suppose that the ray passes symmetrically through the prism, that is, that the angles of incidence and emergence are equal. The angle of incidence i is greater than the angle of refraction r, since the index of the prism is greater than that of air. Hence, as was shown on page 14, when i increases, r also increases, but less rapidly than i. Since in the triangle LRQ the angle L remains constant, then when r increases, r' must undergo a corresponding decrease, for the sum of the three angles of the triangle must be 180 degrees. From the equation $\sin e = n \cdot \sin r'$, it follows that if r' decrease, e must decrease more rapidly than r'. Therefore, if we start with the ray which passes symmetrically through the prism, and increase the angle of incidence, the effect will be to increase the deviation at the first face of the prism and to diminish it at the second face; but as r is now greater than r', the increase at the first face outbalances the decrease at the second face, and the total deviation is increased. If we trace this ray backward, we see the effect of making the angle of incidence smaller

than that of the symmetrical ray, that is, in this case also the deviation is increased. Hence the symmetrical ray is the ray which undergoes the least deviation; it is called the ray of **minimum deviation.**

If D denote the deviation of a ray in passing through a prism, $D = i + e - a$, from which $e = a + D - i$.

We have seen also that $a = r + r'$, from which $r' = a - r$.

Substituting these values of e and r' in the equation

$$\frac{\sin e}{\sin r'} = n,$$

we have
$$\frac{\sin (a + D - i)}{\sin (a - r)} = n,$$

or $\sin [(a + D) - i] = n \cdot \sin (a - r)$, or $\sin (a + D) \cos i - \cos(a + D) \sin i = n (\sin a \cos r - \cos a \sin r)$.

When the angle of the prism is small, $\sin (a + D)$ and $\sin a$ do not differ materially from the angles $a + D$ and a.* Likewise, it is easily seen that the *cosines* of these small angles do not differ materially from unity. Making these substitutions, we have

* The measurement of an angle is expressed by the subtending arc divided by the radius of this arc; it is readily seen that this is practically equivalent to the sine of the angle when the angle is very small.

$$(a + D) \cos i - \sin i = n \cdot a \cos r - n \cdot \sin r;$$

or,

$$D \cdot \cos i = a(n \cdot \cos r - \cos i), \text{ since } \sin i = n \cdot \sin r.$$

Hence $$D = a\left(\frac{n \cdot \cos r}{\cos i} - 1\right).$$

When the ray passes nearly perpendicularly through the prism, as does the ray of minimum deviation in a prism of slight deflecting power, then $\cos r$ and $\cos i$ are both very nearly equal to unity. In this case the deviation is approximately equal to $a(n-1)$. If the index of refraction of the material of which the prism is made is 1.5, as is approximately true of spectacle glass, the deviation becomes equal to $\frac{a}{2}$. The exact index for glass is greater than 1.5; its average index may be regarded as 1.53, and therefore the deviation even in weak prisms is more than one-half the refracting angle of the prism, but for practical purposes the two may be considered equal. In prisms of high deviating power the deviation is perceptibly greater than one-half the refracting angle.

Prisms in the oculists' trial case are usually numbered in degrees of the refracting angle, or according to the deviating power in the position of minimum deviation, the latter method having been first sug-

gested by Dr. Edward Jackson,* of Philadelphia, as being more scientific than the old notation in degrees of the refracting angle. Other systems of numbering prisms have also been advocated and are to some extent used. The units in these systems are: the centrad, introduced by Dennett; † the prism-dioptre, introduced by Prentice; ‡ and the metre-angle, introduced as a measure of convergence by Nagel, and as a prism unit by Maddox.§

It is sometimes desirable to know the result of combining two prisms whose edges are not parallel. We know that prisms deviate light in a direction at right angles to the edge of the prism. Hence when two prisms are combined so that their edges are not parallel, the deviating power of the second prism must be added to that of the first; but the direction in which the power is exerted is not the same in the two prisms.

In Fig. 8, let AB and AD represent the directions in which light is deviated by the first and second prisms respectively; also let the length AB represent the displacement of a ray of light which the first prism produces, as measured on a screen at a fixed distance from the prism, and let AD

* *Trans. American Ophth. Soc.*, 1887 and 1888.

† *Ibid.*, 1889.

‡ *Archives of Ophthalmology*, 1890.

§ Wood's "Med. and Surg. Monographs," Vol. IX., No. 2.

represent the displacement which the second prism produces at the same distance. Then to find the combined effect of the two prisms, it is only necessary to construct the parallelogram *ABCD*, and the diagonal *AC* will represent the direction and the length of displacement produced by the two prisms acting together; for the problem is the same

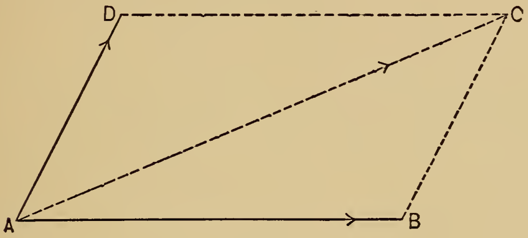

FIG. 8.

as that in which an object at *A* is acted upon by a force which would move it from *A* to *B*, and at the same time by a force which would move it from *A* to *D*, the result being that the object is moved from *A* to *C*.

It would, however, be inconvenient to find the linear displacements *AB* and *AD*; and to avoid this we must use a relation which exists between the angular deviation and the linear displacement.

Let d represent the angle of deviation of the first prism, and d' that of the second prism; then, since

an angle is measured by its subtending arc divided by the radius of this arc, we have

$$d = \frac{\text{arc}}{r} \text{ and } d' = \frac{\text{arc}'}{r}.$$

If the distance of the screen from the prism be the radius, then the arc described with this radius and terminating in A and B will be the subtending arc of the angle d. When d is small, the arc and the straight line AB will be so nearly equal that we may consider them identical. Similarly the arc of the angle d', when this angle is small, may be replaced by the line AD. Thus for small angles we may with any unit lay off AB so that it contains as many units as the arc of the angle d contains degrees, and with the same unit lay off AD so that it contains as many units as the arc of the angle d' contains degrees; then AC as measured by the same unit will give the number of degrees of deviation produced by the equivalent prism, and AC will represent the direction of deviation of this prism. The angle DAB is the angle of inclination of the two prisms, for BA and AD are perpendicular to the edges of the first and second prisms respectively. The angle ABC is equal to $180 - DAB$. Hence in the triangle ABC we know the two sides AB and BC and the included angle ABC. We can find the third side AC and the

angle CAB, which the equivalent prism must make with the first prism.

The side AC is obtained from the equation

$$(AC)^2 = (AB)^2 + (BC)^2 - 2\,AB \times BC \cos ABC.$$

Having found AC, the angle CAB is found from the equation

$$\frac{\sin CAB}{\sin ABC} = \frac{BC}{AC},$$

since in any triangle the sines of the angles are proportional to the opposite sides. If the two prisms are at right angles, then

$$(AC)^2 = (AB)^2 + (BC)^2, \text{ and } \tan\ CAB = \frac{BC}{AB}.$$

CHAPTER II

REFRACTION AT SPHERICAL SURFACES

Let O (Fig. 9) be the centre of a spherical surface, separating two media whose refractive indices are respectively n and n'.

A ray of light, PR, meets the surface at R, and is refracted so as to assume the direction RQ. NRP is the angle of incidence, and ORQ is the angle of refraction. Then we have $n \cdot \sin NRP = n' \cdot \sin ORQ$.

From the triangles ORP and ORQ we have

$$\frac{\sin (180 - NRP)}{\sin ROA} = \frac{\sin NRP}{\sin ROA} = \frac{OP}{RP}. \qquad (1)$$

$$\frac{\sin ORQ}{\sin (180 - ROA)} = \frac{\sin ORQ}{\sin ROA} = \frac{OQ}{RQ}. \qquad (2)$$

Dividing (1) by (2), we have

$$\frac{\sin NRP}{\sin ORQ} = \frac{OP}{RP} \times \frac{RQ}{OQ}. \qquad (3)$$

The first member of (3) is equal to $\frac{n'}{n}$.

Therefore $$n \cdot \frac{OP}{RP} = n' \cdot \frac{OQ}{RQ}.$$

The ray PA is normal to the spherical surface at A; hence it will undergo no deviation, but will continue in the same straight line, PAQ.

This line is called the **axis** of the refracting surface. The ray which meets the surface at R meets the axis at Q. If every other ray of the pencil which is refracted at the spherical surface should meet the axis at Q, this would be the **focus** of the pencil after refraction. A study of Fig. 9, however, will show

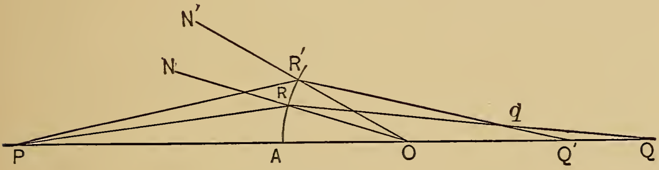

FIG. 9.

that this will not be so. If the arc AR be revolved around the axis, a portion of a sphere will be generated, and all rays meeting this surface at the distance AR from the axis will meet at Q. If we take the rays which meet the spherical surface at a distance AR', they will not meet the axis at Q. The relations between ROA and the angles of incidence and refraction are such that as ROA increases, OQ must diminish; and consequently those rays which meet the refracting surface more remotely from A inter-

sect the axis at a less distance from A than do those rays which lie near the axis.*

The rays RQ and $R'Q'$ meet at q. The curve formed by the locus of all such intersecting points is called the **caustic** of the refracting surface. An illustration of a caustic can be seen by placing a glass of water on a table so that sunlight strikes the glass. The brightly illuminated curve which will be noticed represents the intersection of rays after refraction by the water. The property of spherical refracting surfaces, by which rays of a pencil proceeding from a point meet the axis at different points after refraction, is called **spherical aberration.** The nearer that $\frac{PR}{RQ}$ approaches a constant quantity, the less is the spherical aberration. It is easily seen that this condition is best fulfilled when R is near the axis, and when the curvature of the surface is slight as compared with the distances PA and AQ. The greater the lengths PA and AQ, the farther may R be removed from the axis without appreciable spherical aberration. Spectacle lenses, being lenses of long focal length, do not cause much aberration. In the eye the focal distances are short; but, owing to the cutting off by the iris of all peripheral rays, and to the peculiar construction of the crystalline lens, aberration is not appreciable.

Since the more peripheral rays are too strongly

* Heath's "Geometrical Optics," 2d ed., p. 144.

refracted in comparison with those near the axis, it is evident that the curvature of the spherical surface is too great for peripheral rays. A surface with diminishing curvature, such as an ellipsoid, would consequently produce less aberration than the spherical surface. The curve which produces no aberration is known to mathematicians as the **Cartesian oval.** Such surfaces, however, are not in practical use; for, by suitable combinations of lenses, aberration can be almost entirely overcome, even in the lenses of very short focal distance and wide aperture used in microscopes.

Besides spherical aberration, we have also **chromatic aberration**, which is due to the unequal deviating power of the refracting surface for different colors. In the construction of microscopes and other delicate optical instruments, the annulment of this defect is a matter of the utmost importance, but owing to the long focal length of spectacle lenses and to the prevention by the iris of all peripheral rays from entering the eye, chromatic aberration does not attract attention in ordinary vision.

All our formulæ will be based upon the assumption that PA, RP, AQ, and RQ are great in comparison with the curvature of the surface, and that only rays near the axis are allowed to pass into the refracting medium. With this understanding, our equation,

$$n \cdot \frac{OP}{PR} = n' \cdot \frac{OQ}{RQ},$$

becomes

$$n \frac{(PA + AO)}{PA} = n' \frac{(AQ - AO)}{AQ}.$$

For, upon our assumption, $\frac{OP}{PR}$ will not differ materially from $\frac{OP}{PA}$, and $\frac{OQ}{RQ}$ will not differ materially from $\frac{OQ}{AQ}$.

If PA be denoted by f, and AQ by f', and r be the radius of the spherical surface, the equation will become

$$n \frac{(f+r)}{f} = n' \frac{(f'-r)}{f'}.$$

This equation may be reduced to the form,

$$\frac{n}{f} + \frac{n'}{f'} = \frac{n' - n}{r}. \qquad (a)$$

This is the relation between a point P and its focus Q. P and Q are called **conjugate foci**, and f and f' are **conjugate focal distances.** A pencil of light from P will be brought to a focus at Q, and, conversely, a pencil proceeding from Q will be brought to a focus at P.* In our demonstration we

* When Q is virtual, we must modify this clause so as to read: "A pencil directed towards Q will be focused at P."

have considered all the quantities as positive. This is the convention of signs most convenient in dealing with lenses; but in working out formulæ for several refracting surfaces, it would lead to confusion. In these cases it is better to consider all quantities positive when measured from left to right, and negative when measured from right to left. Thus, AP, measured from A, would be negative; while AQ, as in the former convention, would be positive. We should have to replace f in our equation (a) by $-f$. Doing this, we have

$$\frac{n}{f}-\frac{n'}{f'}=\frac{n-n'}{r}. \qquad (b)$$

We shall use (a) in demonstrating the properties of single lenses, but (b) will be more suitable in tracing the path of light through the several refracting media of the eye.

$$\frac{n}{f}+\frac{n'}{f'}=\frac{n'-n}{r}$$

is the equation between conjugate foci for all positions of f and f'. If, in this equation, we make f infinite, we have

$$f'=\frac{n'r}{n'-n}.$$

This means that if the rays proceed from a point at an infinite distance, that is, if the rays are parallel

to the axis in the first medium, the value for f' is $\frac{n'r}{n'-n}$.* This is called the **posterior** or **second principal focal distance**, and the point where the rays meet the axis is called the **posterior** or **second principal focus**. Similarly, if f' is infinite, that is, if rays proceeding from a point, P, are parallel after refraction, we have, as the corresponding value of f, $\frac{nr}{n'-n}$. The point P thus becomes the **first** or **anterior principal focus**, and the distance $\frac{nr}{n'-n}$ is the **first principal focal distance**. The anterior focal distance is denoted by the letter F, and the posterior focal distance is denoted by F'.

If we divide equation (a) by $\frac{n'-n}{r}$, we have

$$\frac{nr}{(n'-n)f} + \frac{n'r}{(n'-n)f'} = 1,$$

or

$$\frac{F}{f} + \frac{F'}{f'} = 1. \qquad (d)$$

This equation gives us the relation between two conjugate points and the two principal foci.

F and f are positive when to the left of the refracting surface, and F' and f' are positive when to the

* In the small pencils which enter the eye, the rays may be considered parallel when they proceed from a point situated 6 metres from the eye; hence f is infinite when it has not less than this length.

right of this surface. When n' is greater than n, F and F' are both positive or both negative according as r is positive or negative; and when n' is less than n, F and F' are both positive or both negative according as r is negative or positive. In other words, F and F' are both positive when the convex surface is turned toward the medium of less refractive index, and both negative when the convex surface is turned toward the medium of greater index.

We have seen that rays which are parallel to the axis in the first medium will, after refraction into a denser medium at a convex surface, meet in an actual focus in the second medium; and we have also seen that F' in this case is positive. On the other hand, if the refracting surface be concave, the rays will not meet in a point in the second medium; they will be rendered divergent by the refraction, and their direction will be such that, if prolonged backward, they will meet in a point in the first medium. Such a focus is only *imaginary*, not a real meeting point of the rays, as in the former case; it is called a **virtual focus.** It is, as we have seen, negative. Hence, with the convention of signs which we are using, real foci are positive, and virtual foci are negative. Figure 10 illustrates the virtual focus.

A study of equation (d) will show that if n' be greater than n, and r be positive, then, so long as f is greater than F, f' will be positive; and if f be less

than F, f' will be negative. This indicates that if rays proceeding from a point on the axis are refracted into a denser medium at a convex surface, they will after refraction meet the axis in a real point so long as the point from which they proceed is farther from the surface than the first principal focus. If the point is within this focus they will

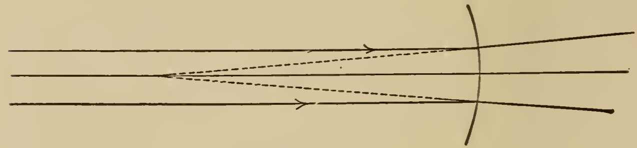

FIG. 10.

not meet the axis after refraction, but if prolonged backward they will meet in a virtual focus, and this will be farther from the surface than the point from which the light proceeds.

The following equations will also be found useful in our studies :

Since $$F = \frac{nr}{n' - n} \quad \text{and} \quad F' = \frac{n'r}{n' - n},$$

we have $$F : F' = n : n'; \text{ or, } \frac{F}{F'} = \frac{n}{n'}.$$

Also, if P (Fig. 11) be the position of a point and Q its conjugate, F and F' the principal foci, then

$PF = f - F$ and $QF' = f' - F'$. Let PF be denoted by u and QF' by u'; then $f = u + F$ and $f' = u' + F'$.

Hence, $$\frac{F}{u+F} + \frac{F'}{u'+F'} = 1.$$

From which we deduce

$$uu' = FF'.$$

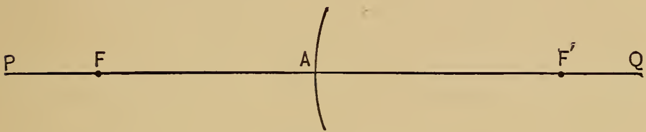

FIG. 11.

Let us next take a point P not on the axis AA'. Draw POP' (Fig. 12) through the centre, O, of the refracting surface. This line is the new axis and P' is the focus conjugate to P. If OP be equal to OA,

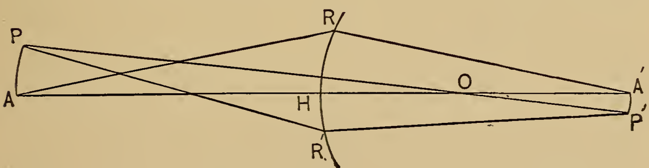

FIG. 12.

then OP' and OA' will be equal, and PA and $P'A'$ will be arcs of circles. As RHR' is revolved on the axis, producing the spherical surface, so the arcs PA and $P'A'$ also produce portions of spherical surfaces.

If the radii OP and OP' be great as compared with the length of these arcs, the portions of spherical surface which they generate will not differ materially from plane surfaces.

Therefore the planes at A and A' perpendicular to the axis AA' are called **conjugate focal planes.** The image of an object lying in the plane at A will lie in the plane at A'. A plane tangent to the refracting surface at H is called the **principal plane** of the refracting surface. If the lines AR and PR' were terminated in such a plane, and from their points of intersection with the plane lines were drawn to A' and P', respectively, the resulting lines would very nearly coincide with those as drawn in Fig. 12; and, for practical purposes, the two results might be considered identical.

The point H, where the principal plane and axis intersect, is called the **principal point.**

We shall now show how we may construct the image of an object if we know the position of certain points. Let PA (Fig. 13) be the linear dimension of an object. If we can determine the point of intersection after refraction of two rays from P, this point will evidently be the image of P. First we take a ray PFR', which passes through the anterior focus F; then we know that this ray must after refraction be parallel to the axis AA'. It is represented by $R'P'$. Next we take a ray PR, which before refrac-

tion is parallel to the axis; then after refraction it must pass through the second principal focus F'. It is represented by RP'. Then P', the point of intersection of the two rays from P, is the image of P. The image of all points in the line AP must lie in the line $A'P'$; hence, $A'P'$ is the image of AP.

We can now appreciate the importance of these points and planes. They are called the **cardinal**

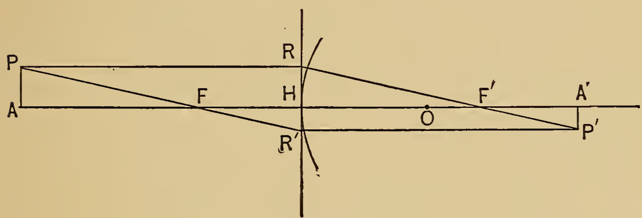

FIG. 13.

points and **planes** of the refracting surface. The centre of the refracting surface also possesses a distinctive property, in that rays which pass through it undergo no deviation, since such rays are normal to the refracting surface. It is called the **nodal point.** The cardinal points of a single refracting surface are four; namely, the *principal point*, the two *principal foci*, and the *nodal point*. The cardinal planes are the *principal plane* and the two *principal focal planes*.

If o be the linear dimension of the object, and i

that of the image, then from the similar triangles PAF and HFR' (Fig. 13) we shall have

$$\frac{PA}{HR'} = \frac{AF}{FH}; \text{ or, } \frac{o}{-i} = \frac{u}{F}; \text{ or, } \frac{o}{i} = -\frac{u}{F}.^*$$

Also from the triangles RHF' and $P'A'F'$,

$$\frac{o}{-i} = \frac{F'}{u'}; \text{ or, } \frac{o}{i} = -\frac{F'}{u'}.$$

From either of these equations we can determine the size of the image. We may also determine the size of the image in terms of the divergence of the

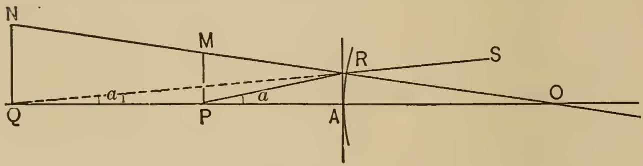

FIG. 14.

rays before and after refraction. Let MP (Fig. 14) be the linear dimension of an object. A ray from P, meeting the refracting surface at R, would be refracted so as to assume the direction RS; and if prolonged backward, it would meet the axis at Q, which is conjugate to P. Likewise, N is conjugate to M, and QN is the image of PM. The angle

* Since the image, when real, as in the figure, lies on the opposite side of the axis to the object, we must consider it negative.

RPA, which we call a, expresses the divergence of the pencil before refraction, and RQA or a_1 is the divergence after refraction.

Then $$\tan a = \frac{AR}{PA}, \text{ and } \tan a_1 = \frac{AR}{QA};$$

from which $$\frac{\tan a}{\tan a_1} = \frac{QA}{PA} = \frac{f'}{f}.$$

We have also the equation

$$\frac{PO}{QO} = \frac{PM}{QN}.$$

From the law of refraction we have, as on page 26,

$$n \cdot \frac{PO}{RP} = n' \cdot \frac{QO}{RQ},$$

or, since RP may for rays near the axis be replaced by f and RQ by f',

$$n \cdot \frac{PO}{f} = n' \cdot \frac{QO}{f'}, \text{ or } \frac{PO}{QO} = \frac{n'f}{nf'};$$

hence $$\frac{PM}{QN} \text{ or } \frac{o}{i} = \frac{n'f}{nf'}, \text{ and } o \cdot \frac{n}{f} = i \cdot \frac{n'}{f'}. \quad (1)$$

From this equation and from $\frac{f'}{f} = \frac{\tan a}{\tan a_1}$ we deduce

$$o \cdot n \cdot \tan a = i \cdot n' \cdot \tan a_1. \quad (2)$$

This is known as Helmholtz' formula.

CHAPTER III

REFRACTION THROUGH LENSES

Having investigated the refraction of light at one spherical surface, we are now prepared to study refraction through lenses. A **lens** is defined as a portion of a refracting substance bounded by two curved surfaces centred on the same axis. If the radius of curvature of one surface is infinite, then the corresponding surface is plane, and the lens is bounded by one curved and one plane surface. Ordinarily, the surfaces of lenses are spherical; but lenses have been constructed whose surfaces were ellipsoidal or paraboloid. The only lenses in practical use, however, are those whose generating curve is a circle; and we shall confine our attention entirely to lenses of this nature.

The **thickness** of a lens is the distance between the bounding surfaces as measured along the axis.

A lens bounded by two convex surfaces is called a **double convex** or **bi-convex** lens; one bounded by two concave surfaces is called a **bi-concave** lens.

A lens, of which one surface is convex and one concave, is called a **convexo-concave** lens, or a **menis-**

cus. Lenses of this form are used as spectacles, and are known as **periscopic** lenses.

The terms **plano-convex** and **plano-concave** need no explanation.

In our demonstration we shall take as the typical case the double convex lens; and we shall consider the refractive index of the material of which the lens is composed to be greater than that of the air by which it is surrounded. The index of the lens will be indicated by the letter n, and that of the air by unity.

We shall first show that there are two points on the axis of the lens which are useful in the determination of the positions of conjugate foci. These points are a pair of conjugate foci, such that any incident ray directed toward one of them will, after refraction, appear to come from the other, and in a direction parallel to that before refraction. These points are called the **nodal points** of the lens.

To find the position of these points, we draw any radius of the first surface, as OQ (Fig. 15). Next we draw a radius, $O'Q'$, of the second surface, so that OQ and $O'Q'$ are parallel. Connect the points Q and Q' by the straight line QQ', which meets the axis at C. Then from the similar triangles OCQ and $O'CQ'$, we have $OC : O'C = OQ : O'Q' = r : r'$,

from which
$$\frac{OC}{O'C} = \frac{r}{r'}.$$

Therefore, $\frac{OC}{O'C}$ is a constant quantity, irrespective of the position of Q, from which the first radius is drawn; and consequently C must be a fixed point. The ray of light, RR', which passes through C, is, after refraction, parallel to its direction before re-

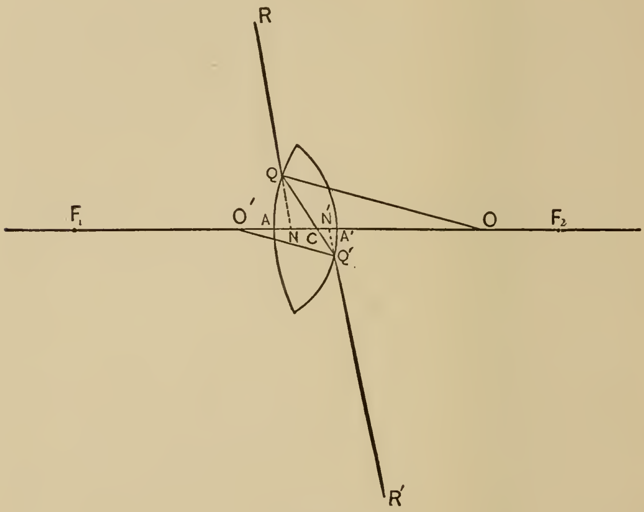

FIG. 15.

fraction; because, the radii OQ and $O'Q'$ being parallel, the planes perpendicular to the curved surfaces at Q and Q' are parallel, and the lens will, for this ray, act as would a piece of plane glass.

Let N be the point on the axis toward which the ray is directed before refraction. The ray is so re-

fracted at the first surface as to pass through C; then C is conjugate to N in the first refraction. After refraction at the second surface, the ray appears to pass through N'; then N' is conjugate to C in the second refraction. Since N is the *virtual* intersecting point of the ray and axis before refraction, and N' is the *virtual* intersecting point of the ray and axis after refraction, then N and N' are conjugate points with respect to both refractions.

The point C is called the **optical centre** of the lens. All rays which pass through it are, after refraction through the lens, parallel to their direction before refraction.

To find the position of the points N, N', and C we have the following relations :

$$OO' = r + r' - e\,;$$

where r and r' are the radii of the surfaces, and e is the thickness of the lens. Also,

$$\frac{OC}{O'C} = \frac{r}{r'};$$

and consequently,

$$\frac{OC}{OC + O'C} = \frac{r}{r + r'}\,; \text{ or, } \frac{OC}{r + r' - e} = \frac{r}{r + r'},$$

$$OC = \frac{r}{r + r'}(r + r' - e) = r - \frac{er}{r + r'},$$

$$AC = OA - OC = r - r + \frac{er}{r+r'} = \frac{er}{r+r'}.$$

Similarly, $$A'C = \frac{er'}{r+r'}.$$

But since in the first refraction N and C are conjugate foci, we have from formula (a), page 30,

$$\frac{1}{AN} + \frac{n}{AC} = \frac{n-1}{r}.$$

Substituting the value of AC, this becomes

$$\frac{1}{AN} + \frac{n(r+r')}{er} = \frac{n-1}{r}.$$

If rays proceeding from F_1 are parallel to the axis after refraction at the first surface, and these parallel rays after refraction at the second surface meet at F_2, then

$$AF_1 = \frac{r}{n-1} = F_1, \text{ and } A'F_2 = \frac{r'}{n-1} = F_2.$$

Making these substitutions, we have

$$\frac{1}{AN} = -\frac{n(F_1+F_2)}{eF_1} + \frac{1}{F_1};$$

or, $$AN = -\frac{F_1}{n(F_1+F_2)-e} \times e.$$

If e be replaced by $n \cdot c$, this equation will become

$$AN = -\frac{cF_1}{F_1 + F_2 - c}.$$

Since AN is negative for the convex lens, N lies to the right of A, and since in the lenses which are in practical use, the numerator F_1 is less than the denominator $n(F_1 + F_2) - e$, the length AN is less than e, and the nodal point N lies within the lens. Similarly, we find that the nodal point N' lies within the convex lens, and that its distance from A' is

$$\frac{cF_2}{F_1 + F_2 - c}.$$

Using the same equation, but changing the sign of r, we should find that in bi-concave lenses, also, the nodal points lie within the lens.

To repeat, the *nodal points* are two conjugate points such that a ray of light directed toward one of them is so refracted as to pass through the *optical centre* of the lens, and emerges in a direction *parallel to that before refraction.* Such a ray undergoes only a lateral displacement due to the thickness of the lens.

We shall now demonstrate the method of finding the focus Q, after refraction by a lens, of a pencil of light from a point P on the axis of the lens. The ray PR (Fig. 16) meets the first refracting

surface at R; it is refracted toward R', and, if prolonged, it would meet the axis at Q'. But, after travelling the distance RR', it meets the second

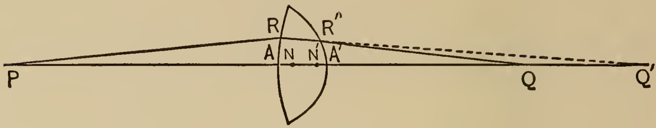

FIG. 16.

surface of the lens and is refracted to Q. Then in the first refraction Q' is conjugate to P, and we have from equation (a), page 30,

$$\frac{1}{PA}+\frac{n}{AQ'}=\frac{n-1}{r};$$

or, substituting f for PA, f' for AQ', and F_1 for $\frac{r}{n-1}$,

$$\frac{1}{f}+\frac{n}{f'}=\frac{1}{F_1}. \tag{1}$$

Also in the second refraction Q is conjugate to Q'; therefore,

$$\frac{n}{A'Q'}+\frac{1}{A'Q}=\frac{n-1}{r'};$$

or, substituting f_1' for $A'Q'$, f_1 for $A'Q$, and F_2 for $\frac{r'}{n-1}$, we have

$$\frac{n}{f_1'}+\frac{1}{f_1}=\frac{1}{F_2}. \tag{2}$$

Reference to the figure will indicate that AA', the thickness of the lens, is equal to the difference in length between f' and f_1'; but we also notice that f' and f_1' have opposite signs, that when f' is positive, f_1' is negative. Hence the difference between f' and f_1' will be expressed algebraically by $f'+f_1'$; and if e represent the thickness of the lens, $f'+f_1'=e$, or nc, n being the refractive index of the lens, and c being of such value that e is equal to nc. From these equations we can determine the relation between P and Q. From (1) we obtain

$$f' = \frac{n}{\frac{1}{F_1}-\frac{1}{f}}.$$

From (2), $$f_1' = \frac{n}{\frac{1}{F_2}-\frac{1}{f_1}}.$$

Substituting these values in the equation $f'+f_1' = nc$, we have

$$\frac{n}{\frac{1}{F_1}-\frac{1}{f}} + \frac{n}{\frac{1}{F_2}-\frac{1}{f_1}} = nc. \qquad (3)$$

This, by reduction, becomes

$$ff_1(F_1+F_2-c)-F_1f_1(F_2-c)$$
$$-F_2f(F_1-c)=cF_1F_2 \cdots. \qquad (4)$$

This equation is true for all values of f and f_1. If we make f infinite, the corresponding value of f_1 will give us the focus for rays parallel to the axis before refraction.

Dividing equation (4) by f and making f infinite, we derive

$$f_1 = \frac{F_2(F_1 - c)}{F_1 + F_2 - c}.$$

Similarly, if f_1 be infinite, that is, if the rays be parallel to the axis after refraction, we shall obtain

$$f = \frac{F_1(F_2 - c)}{F_1 + F_2 - c}.$$

The points determined by these equations are the *principal focal* points of the lens. They are not usually measured from the surfaces of the lens, however, but from the points N and N'. It will be seen that as thus measured the two focal distances are equal.

$$f + AN = \frac{F_1(F_2 - c)}{F_1 + F_2 - c} + \frac{cF_1}{F_1 + F_2 - c} = \frac{F_1 F_2}{F_1 + F_2 - c},$$

$$f_1 + A'N' = \frac{F_2(F_1 - c)}{F_1 + F_2 - c} + \frac{cF_2}{F_1 + F_2 - c} = \frac{F_1 F_2}{F_1 + F_2 - c}.$$

Hence in a lens the two principal focal distances are equal, and this distance is found from the equation

$$F = \frac{F_1 F_2}{F_1 + F_2 - c},$$

where F_1 and F_2 represent the same quantities as on page 44.

The nodal points N and N' possess another important property in addition to that already demonstrated. If planes be drawn through these points perpendicular to the axis of the lens, an object in the first plane will have its image in the second plane, and the image and object will be of the same size and on the same side of the axis; in other words, the line joining the points where the incident and refracted rays meet, respectively, the first and second planes, is parallel to the axis. The two planes are called the **principal** planes, and the points where they meet the axis are called the **principal** points. In lenses, the principal points and nodal points coincide; but this is not so in all optical systems, as we shall hereafter learn.

To prove this property of the principal planes, let o be the linear magnitude of an object, i its image after one refraction, and i' its image after two refractions.

Then from equation (1), page 39, we have

$$\frac{o \times 1}{f} - \frac{n \cdot i}{f'} = 0, \quad \frac{i \cdot n}{f_1'} - \frac{i'}{f_1} = 0,$$

$$\frac{o}{i} = \frac{nf}{f'}, \quad \text{and} \quad \frac{i'}{i} = \frac{nf_1}{f_1'}.$$

Hence, $$\frac{o}{i'} = \frac{ff_1'}{f'f_1}.$$

But in demonstrating the properties of nodal points we have seen that $AC : A'C = r : r'$. Since in the first refraction C is conjugate to N, then AN is represented by f and AC by f'; and since in the second refraction N' is conjugate to C, then $A'C$ is represented by f_1' and $A'N'$ by f_1. Hence,

$$\frac{f'}{f_1'} = \frac{r}{r'}.$$

Applying formula (a), page 30, to the first refraction, we have

$$\frac{1}{f} + \frac{n}{f'} = \frac{n-1}{r}; \text{ or, } \frac{f' + nf}{ff'} = \frac{n-1}{r}. \qquad (1)$$

Similarly, applying this formula to the second refraction,

$$\frac{1}{f_1} + \frac{n}{f_1'} = \frac{n-1}{r'}; \text{ or, } \frac{f_1' + nf_1}{f_1 f_1'} = \frac{n-1}{r'}. \qquad (2)$$

Dividing (2) by (1), we derive

$$\frac{ff'(f_1' + nf_1)}{f_1 f_1'(f' + nf)} = \frac{r}{r'} = \frac{f'}{f_1'},$$

from which

$$\frac{f(f_1' + nf_1)}{f_1(f' + nf)} = 1;$$

or, $\quad ff_1' + nff_1 = f_1 f' + nff_1$, and $ff_1' = f_1 f'$.

Therefore the equation $\frac{o}{i'} = \frac{ff_1'}{f'f_1}$ becomes $o = i'$, which shows that object and image are equal when they lie in the planes perpendicular to the axis at the nodal points of the lens. The significance of the principal planes is rendered clear by reference to Fig. 17. A thorough understanding of the properties of these planes is necessary to the further study

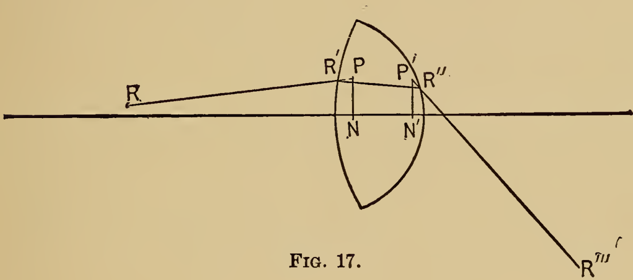

FIG. 17.

of refraction. A ray of light RR' meets the first surface at R'. It is directed toward P, but before reaching P, it is refracted at R', so as to assume the direction $R'R''$; at R'' it is again refracted so that its direction on leaving the lens is $R''R'''$. If $R''R'''$ be prolonged backward, it will meet the plane $P'N'$ at P', and PN and $P'N'$ will be equal. Any other ray directed toward P will after refraction appear to come from P'; in other words, P' is the image of P. It must not be supposed, however, that a real object placed in the lens substance at PN would have for

its image $P'N'$; for such an object, as seen by an eye, would have undergone refraction at one surface of the lens only. PN and $P'N'$ are both *virtual.*

By making use of the properties which belong to the principal planes, we can construct the image of an object after refraction through a lens. Let PA (Fig. 18) be the linear dimension of an object, F and F' the principal foci of the lens, H and H' the

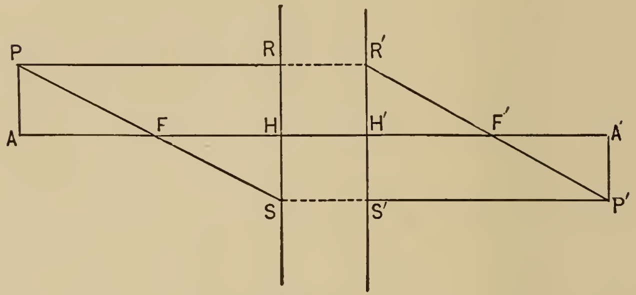

FIG. 18.

principal points, RS and $R'S'$ the principal planes. From the point P draw a ray PFS; then since it passes through the anterior focus, it must after refraction be parallel to the axis, and from the property of principal planes just demonstrated HS and $H'S'$ must be equal. Hence $S'P'$ represents the ray. Next, take a ray PR from P, parallel to the axis; then it must after refraction pass through the posterior focus F', and HR and $H'R'$ are equal.

$R'F'P'$ will represent the ray after refraction, and P', the point of intersection of the two rays from P, will be the image of P. Similarly, we can show that any other point of PA has a corresponding image in $P'A'$, and $P'A'$ is the image of PA. To find the size of the image we use the similar triangles PAF and FHS, or $F'A'P'$ and $H'R'F'$. When the image is real, as in our figure, it lies on the opposite side of the axis to PA, and it must be considered negative. As on page 35, the distance of the object from the first principal focus is denoted by the letter u, and the distance of the image from the second focus is denoted by u'. Hence, we have the following equations:

$$\frac{PA}{HS}=\frac{AF}{FH}; \text{ or, } \frac{o}{-i}=\frac{u}{F}.$$

Also,
$$\frac{P'A'}{H'R'}=\frac{F'A'}{H'F'}; \text{ or, } \frac{o}{-i}=\frac{F}{u'}.$$

By referring to page 38, we see that these equations are the same as those which we found to determine the size of the image after one refraction; but, instead of one principal plane from which all distances are measured, we now have two such planes.

Having solved the problem for the double convex lens, we may, without repeating the investigation, apply the same formulæ to other lenses by making suitable changes in the signs of radii and foci.

The position of the principal or nodal points may be found from the equations,

$$AN\ (\textit{which we call } h) = \frac{cF_1}{F_1 + F_2 - c},$$

and
$$A'N'\ (\textit{or } h') = \frac{cF_2}{F_1 + F_2 - c}.$$

The focal distance of the lens is found from the equation

$$F = \frac{F_1 F_2}{F_1 + F_2 - c}.$$

If, in these equations, we replace the value of F_1 and F_2 in terms of n and r, we obtain

$$h = \frac{cr}{(r + r') - (n - 1)c}, \qquad (1)$$

$$h' = \frac{cr'}{(r + r') - (n - 1)c}, \qquad (2)$$

$$F = \frac{rr'}{(n - 1)[r + r' - (n - 1)c]}. \qquad (3)$$

In spectacle lenses the thickness is so slight, as compared with the radius of curvature, that it may be ignored. Such lenses are spoken of as **thin** lenses. In thin lenses the nodal points coincide with the centre of the lens; and the equation which determines F reduces to the form

$$F = \frac{rr'}{(n-1)(r+r')}.$$

This equation serves to determine the focal length in all kinds of spectacle lenses. If both r and r' are positive and equal, we have

$$F = \frac{r}{2(n-1)};$$

and, if we reckon n as 1.5 for the glass of which spectacles are made, we have $F = r$. If r' is infinite, that is, if the lens is plano-convex, then

$$F = \frac{r}{n-1} = 2\,r,$$

when n is 1.5. If r and r' are both negative, the lens is bi-concave, and we have

$$F = \frac{rr'}{-(n-1)(r+r')}.$$

Hence the principal focal distances are negative. This means that rays which are parallel to the axis, meeting the lens, do not intersect the axis after refraction; but they would, if prolonged backward, meet the axis on the same side of the lens as that from which the rays proceed. Similarly, rays which are parallel to the axis after refraction, do not come from a real point; but before refraction they are

directed toward a point on the axis and on the opposite side of the lens to that from which the light proceeds. Thus we see that in the concave lens both foci are virtual; also that the first focus lies behind the lens, and the second focus lies in front of the lens, considering, as we do, that the side of the lens

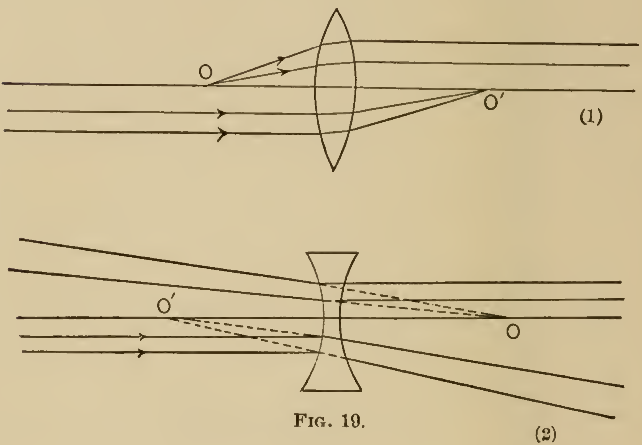

Fig. 19.

which is turned toward incident light is the front of the lens.

In Fig. 19, (1) represents the action of a convex lens, (2) represents the action of a concave lens. All rays from O, the anterior focus of the convex lens, are after refraction parallel to the axis; all rays, which before refraction are parallel to the axis, meet at the posterior focus O'. Convex lenses are

called **collective** lenses. In the concave lens rays which are parallel to the axis before refraction are rendered divergent after refraction. If prolonged backward, they would meet at O'. O' is then the posterior focus of the lens, for it is the virtual intersecting point with the axis of rays which before refraction are parallel to the axis. Similarly, rays converging to the point O before refraction are parallel to the axis after refraction. O is therefore the anterior focus of the lens, since it is the virtual intersecting point with the axis of rays which are parallel to the axis after refraction. Concave lenses are called **dispersive** lenses.

If, in equation (4), page 47, we disregard the thickness of the lens, making c equal to zero, and if we substitute F, the focal distance of the lens, for its equivalent, the equation reduces to the form,

$$\frac{F}{f}+\frac{F}{f_1}=1.$$

Equation (4) can be reduced to the same form, irrespective of the value of c, but not so simply as when c is equal to zero; and, as we need apply this formula to thin lenses only, we shall not make the substitution for thick lenses. From the expression thus obtained, we see that after refraction by convex lenses, conjugate foci lie on opposite sides of the lens and are real so long as both conjugate points

are without the principal foci, but if one point is within the principal focus, and real, the other lies without the principal focus, on the same side of the axis as the first point, and is virtual. We also observe from a study of this equation that as a point approaches the lens, its conjugate moves in the same direction, that is, it recedes from the lens when it is real, and approaches the lens when it is virtual. Since an object lying in one conjugate plane has its image in the other, we may apply the foregoing conclusions to the images of objects. Since, as we have seen, real images are inverted, a real and inverted image of an object will be formed by a convex lens, provided the object be not within the principal focus of the lens. If the object be placed at the principal focus of the lens, the rays from every point of the object will be rendered parallel, and no image will be formed. If the object be placed within the principal focus, the rays from every point of the object will, after refraction, be divergent, and if prolonged backward will meet in a virtual focal plane, thus forming a virtual image.

If, in the equation which we have been studying, we make F negative, as in the concave lens, we shall see that rays from an object will not, after refraction, meet in a real focus.

Since real images are formed by the actual intersections of rays of light, they may be depicted upon

a screen, or upon a sensitive photographic plate which is capable of retaining the impression; or, again, upon the retina of the eye, a nervous mechanism through which the impression is transmitted to the brain, where it is manifested as vision. Virtual images, not being actual intersecting points of rays, cannot be so depicted.

Plano-convex lenses act in the same manner as bi-convex lenses; and plano-concave act as bi-concave lenses. Periscopic lenses act as convex or concave lenses according as the convex or concave surface has the greater curvature. Reference to the equation

$$F = \frac{rr'}{(n-1)(r+r')}$$

renders this apparent.

The power of a lens is inversely proportional to the focal length. If F represents the focal length, $\frac{1}{F}$ is the power of the lens. Lenses may be numbered according to their focal length or according to their power. A lens which has a focal length of ten inches is called a *ten-inch* lens. Its power is expressed by $\frac{1}{10}$. Since, in lenses which have equal curvature at the two surfaces and whose index is 1.5, the focal length is equal to the radius of curvature, the power of the lens is expressed by $\frac{1}{r}$.

We obtain the power of two or more lenses used in combination by adding the expressions denoting the power of each lens; but this simple method applies only when the distance between the centres of the lenses is so slight that it may without error be neglected. That the power of two or more lenses used in combination is equal to the sum of the powers of the lenses, may be proved from the relation between bi-curved and plano-curved lenses. Since the principal or nodal points and the optical centre of a thin lens coincide, any bi-curved thin lens is equivalent in all respects to a plano-curved lens of the same focal length. Hence if we wish to combine two lenses, we may first replace them by two plano-curved lenses with their plane faces in contact. We then have as the result of this combination a bi-curved lens; and the focal length of this lens is found from the equation

$$F = \frac{F_1 F_2}{F_1 + F_2 - c}.$$

When the thickness of the lens is neglected, this becomes

$$F = \frac{F_1 F_2}{F_1 + F_2},$$

F_1 being the focal length of the first, and F_2 that of the second of the component lenses.

From this we find

$$\frac{1}{F}=\frac{1}{F_1}+\frac{1}{F_2}.$$

To find the power obtained by combining a lens of ten inches focal length with one of twenty inches, we have

$$\frac{1}{F}=\frac{1}{10}+\frac{1}{20}=\frac{3}{20};\text{ or, } F=6\tfrac{2}{3}\text{ inches.}$$

In order to avoid fractions in the addition of lenses, another method of numbering them is employed. This method, which possesses many advantages, has almost entirely displaced the old method in ophthalmology. The unit of power is that of a lens whose focal length is one metre. This unit is called a **dioptre**. A lens which has a focal length of one-half metre has therefore a power of two dioptres. A lens whose focal length is two metres has a power of .5 dioptre, and so on. Hence to find the power of any number of lenses used in combination, we have only to add their dioptric values. Thus a lens of two dioptres in combination with one of three dioptres is equivalent to a lens of five dioptres.

CHAPTER IV

THE EYE AS AN OPTICAL SYSTEM

The eye as an optical system consists of three refracting surfaces and three media. The first surface is the cornea. Strictly speaking, this is not a spherical surface; it conforms more closely to the small end of an ellipsoid of three unequal axes, but we may without appreciable error replace in our calculations the normal cornea by a spherical surface. The anterior and posterior surfaces of the cornea being very nearly parallel, and the refractive index of the cornea and aqueous being practically identical, the cornea may be disregarded and the aqueous humor may be considered the first refractive medium. After traversing the aqueous, light enters the crystalline lens. This is not composed of a homogeneous medium, but of numerous layers, the density of which increases from the outer to the central part of the lens. To trace the path of light through each of these layers would be an impossible task. Helmholtz, accordingly, divides the lens into three portions with increasing index,—the cortical

or outer portion, the intermediate, and the nuclear portion. From these he has determined the mean refractive index of a lens having the same curvature and refractive power as the crystalline lens. Reference to Fig. 20 will show that this does not mean that the refractive index of Helmholtz' equivalent is equal to the mean of the indices of the three portions into which the lens is divided. The index of the equivalent is greater than the greatest index of the component portions of the lens; for if the index of the entire lens were equal to that of the nucleus, its refractive power would be less than in the lens as constituted.

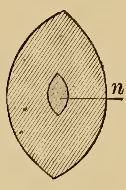

Fig. 20.

This will be seen from the figure. The letter n indicates the nuclear portion, which has a small radius of curvature; and therefore, acting alone, it would have a greater refractive power than the entire lens of the same index, for its curvature is greater. The outer portion has the same effect as if two divergent menisci were added to the nucleus. The index of the outer portion being less than that of the nucleus, the addition of the two menisci has a less divergent effect than if they had the higher index of the nucleus. Thus we see that the refractive power of the lens with increasing index is greater than if it were composed of homogeneous material with the index of the most highly refract-

ing part of the lens; and consequently the index of a homogeneous lens having the curvature and power of the crystalline lens must be greater than that of the nucleus of the lens. This equivalent, as determined by Helmholtz, is 1.4371.

Another effect of this increasing index is to diminish spherical aberration. We have seen that a spherical lens has a greater refractive power for rays that meet it at a distance from the axis than for those which pass near the axis. By means of the physiological arrangement of layers with increasing index, those rays which meet the lens near the axis are acted upon by the more highly refracting nucleus, while those which are remote from the axis escape this portion and are refracted only by the cortical layers of the lens.

Finally, after passing through the lens, light enters the vitreous, whose refractive index is the same as that of the aqueous.

Gauss, the eminent mathematician, has by his researches rendered it possible for us to trace the path of light from an external object through the media of the eye to its focus on the retina.* Prior to his work other mathematicians, notably Moebius, had investigated refraction through a number of media, but they neglected the thickness of the lenses,

* Gauss, "Dioptrische Untersuchungen," Werke, Band V. Göttingen, 1867.

thereby causing an appreciable error in dealing with lenses of considerable thickness in comparison with their focal length, as in the case of the lens of the eye. Furthermore, Gauss demonstrated that an optical system of any number of spherical surfaces and media, the surfaces all being centred on the same axis, has certain points and planes which are very useful in determining the optical effect of the system. These are called **cardinal** points and planes, and are similar to the cardinal points and planes of single lenses. As in single lenses we have:

The first and second *principal points* and the first and second *principal planes*, and

The first and second *principal foci* and the first and second *principal focal planes.*

When in any system the positions of these points and planes have been determined, the solution of the system is complete.

There are also two other points, the first and second *nodal* points, which are similar to the nodal points of a lens. These points, though useful, are not necessary to the solution of the system. Their properties were first demonstrated by Listing.

Let P (Fig. 21) represent a point on the axis of the system; and let HR represent the first surface or cornea of the eye; $H'R'$ the anterior surface and

$H''R''$ the posterior surface of the crystalline lens. A ray of light passing through P meets the cornea at R; it will be refracted so as to assume the direction RR', and if there were no further refraction, it would meet the axis at Q'. But after travelling the distance RR', it meets the anterior surface of the crystalline lens and is refracted so as to assume the direction $R'R''$, and if it continued in this direction, it would meet the axis at Q''. But at the posterior surface of the lens it is again refracted so that its final direction is $R''Q$. Hence we see that in the first refraction Q' is conjugate to P; in the second refraction Q'' is conjugate to Q'; and in the third and last refraction Q is conjugate to Q''. The point Q being the point of intersection of the ray with the axis after the three refractions, is conjugate to P with reference to the entire system.

To find the relation between the position of a point and its conjugate after refraction through the eye, we make use of the following constants:

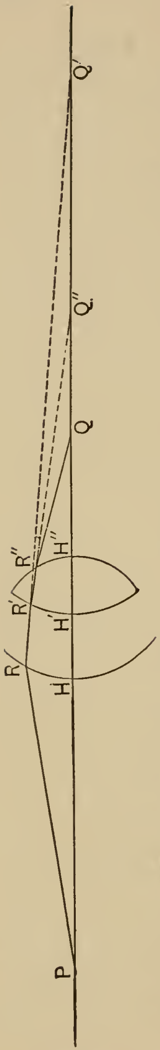

Fig. 21.

The refractive index of air, which is denoted by	1
The refractive index of the cornea and aqueous	n
The refractive index of the crystalline lens . .	n'
The refractive index of the vitreous	n
The radius of curvature of the cornea . . .	r
The radius of curvature of the anterior surface of the lens	r'
The radius of curvature of the posterior surface of the lens	r''
The distance of anterior surface of the cornea from the anterior surface of the lens . . .	nt_1
The thickness of the crystalline lens	$n't_2$

All distances measured from left to right are positive, and those measured from right to left are negative. In accordance with this convention of signs, we use formula (b), page 31. Then r and r' are positive and r'' is negative. The distance from H to H' (Fig. 21) is denoted by nt_1, and that from H' to H'' by $n't_2$. Let the distance of P from the first surface be denoted by f; and the distance of Q' from the same surface by nf_1. The distance of the second image Q'' from the second surface H' is denoted by $n'f_2$, and the distance of the third image Q from the third surface H'' by nf_3.*

* The reason for this notation will become apparent in the course of the demonstration.

Hence

$PH = f$; $HQ' = nf_1$; $H'Q'' = n'f_2$; and $H''Q = nf_3$.

Then at the first refraction, applying formula (b), page 31, we have

$$\frac{1}{f} - \frac{n}{nf_1} = \frac{1-n}{r}. \tag{1}$$

At the second refraction,

$$\frac{n}{H'Q'} - \frac{n'}{n'f_2} = \frac{n-n'}{r}.$$

But $H'Q' = HQ' - HH' = nf_1 - nt_1$; hence

$$\frac{1}{f_1 - t_1} - \frac{1}{f_2} = \frac{n-n'}{r'}. \tag{2}$$

At the third refraction,

$$\frac{n'}{H''Q''} - \frac{n}{nf_3} = \frac{n'-n}{-r''}.$$

But $H''Q'' = H'Q'' - H'H'' = n'f_2 - n't_2$; therefore

$$\frac{1}{f_2 - t_2} - \frac{1}{f_3} = -\frac{n'-n}{r}. \tag{3}$$

For convenience we make

$$\frac{1-n}{r} = k_0;\quad \frac{n-n'}{r'} = k_1;\ \text{ and }\ -\frac{n'-n}{r''} = k_2.$$

From (1) we have

$$\frac{1}{f} = k_0 + \frac{1}{f_1}.$$

From (2), $$f_1 = t_1 + \cfrac{1}{k_1 + \cfrac{1}{f_2}}.$$

From (3), $$f_2 = t_2 + \cfrac{1}{k_2 + \cfrac{1}{f_3}}.$$

Substituting these values of f_1 and f_2, we have

$$\frac{1}{f} = k_0 + \cfrac{1}{t_1 + \cfrac{1}{k_1 + \cfrac{1}{t_2 + \cfrac{1}{k_2 + \cfrac{1}{f_3}}}}}. \tag{4}$$

This is an equation expressing the relation between f and f_3, from which we can find the conjugate focus of a point, P, in any case. The same method may be applied to any number of surfaces, but as the equation becomes much more cumbersome with each additional surface, it is convenient to obtain from the known properties of such an expression a simpler relation between f and f_3. The second member of equation (4) is called a **continuous**

fraction. If we should neglect all the terms to the right of k_0, the expression would become

$$\frac{k_0}{1}. \tag{a}$$

If we neglect all except $k_0 + \frac{1}{t_1}$, we have

$$\frac{k_0 t_1 + 1}{t_1}. \tag{b}$$

Neglecting now all after $k_0 + \cfrac{1}{t_1 + \cfrac{1}{k_1}}$, we have

$$\frac{k_1(k_0 t_1 + 1) + k_0}{k_1 t_1 + 1}. \tag{c}$$

Continuing this process, we finally embrace all the terms and get the true value of the second member of the equation. The expressions (*a*), (*b*), and (*c*) are called **convergents.** If we examine the convergent (*c*), we shall see that its numerator is obtained by multiplying the new letter k_1 by the numerator of the preceding convergent and adding the numerator of the second preceding convergent; and that the denominator is obtained by multiplying the new letter k_1 by the denominator of the preceding convergent and adding the denominator of the second preceding convergent. This relation is true for all subsequent terms of a continuous fraction such as the second member of equation (4); we can there-

fore write out the value of this fraction without further calculation. The successive convergents, as thus obtained, are:

$$\frac{k_0}{1}, \tag{a}$$

$$\frac{k_0 t_1 + 1}{t_1}, \tag{b}$$

$$\frac{k_1(k_0 t_1 + 1) + k_0}{k_1 t_1 + 1}, \tag{c}$$

$$\frac{t_2\{k_1(k_0 t_1 + 1) + k_0\} + k_0 t_1 + 1}{t_2(k_1 t_1 + 1) + t_1}, \qquad \frac{(g)}{(h)}$$

$$\frac{k_2[t_2\{k_1(k_0 t_1+1)+k_0\}+k_0 t_1+1]+k_1(k_0 t_1+1)+k_0}{k_2\{t_2(k_1 t_1+1)+t_1\}+k_1 t_1+1}, \qquad \frac{(k)}{(l)}.$$

For convenience we call the last two of these convergents $\frac{g}{h}$ and $\frac{k}{l}$; then the next expression, which will be the true value of the fraction, may be written $\frac{f_3 k + g}{f_3 l + h}$, and equation (4) becomes

$$\frac{1}{f} = \frac{f_3 k + g}{f_3 l + h}.$$

If we multiply numerator and denominator of the second member of this equation by n, we shall have

$$\frac{1}{f} = \frac{n f_3 k + n g}{n f_3 l + n h},$$

or
$$n f_3 \cdot l + nh = f \cdot n f_3 \cdot k + f \cdot ng. \tag{5}$$

Since nf_3 is the distance of the last image, Q, from the last surface, equation (5) expresses the relation between P and its conjugate Q. This is a general equation, true for all positions of P; we may therefore, by making nf_3 and f respectively equal to infinity, find the positions of the principal foci.

Equation (5) may be written

$$(fk - l)\,nf_3 = nh - f \cdot ng,$$

or

$$fk - l = \frac{nh - f \cdot ng}{nf_3}.$$

When $nf_3 = \infty$, $$f = \frac{l}{k}.$$

Similarly we find when $f = \infty$,

$$nf_3 = -\frac{ng}{k}.$$

These values determine the first and second principal foci as measured respectively from the anterior surface of the cornea and the posterior surface of the crystalline lens; but we cannot construct the image of an object until we find the position of the principal planes. These planes are, as we know, conjugate focal planes such that an object in one of them will have its image in the other, the object and image being equal and on the same side of the axis. Hence to find these planes in a system of refracting

surfaces, it is evident that we may use equation (5), which expresses the relation between conjugate foci. If, in this equation, we impose the condition that object and image be of the same size and sign, the resulting values of f and nf_3 will be the distances of the first and second principal planes from the anterior surface of the cornea and posterior surface of the crystalline lens, respectively.

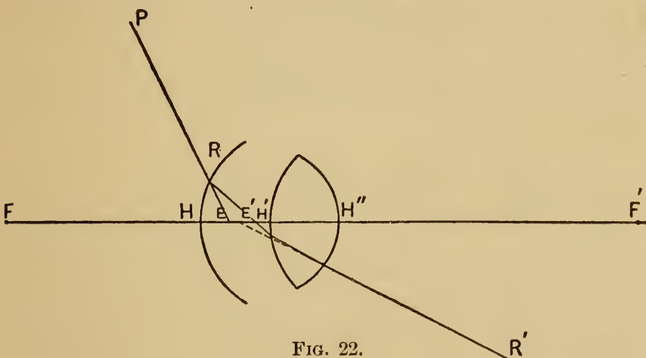

FIG. 22.

In Fig. 22 let PR represent an imaginary ray directed toward E. After refraction through the system it appears to pass through E'; then E and E' are conjugate foci. Let F mark the position of the anterior, and F' the position of the posterior focus of the system; then

$$FH = \frac{l}{k}; \text{ and } H''F' = -\frac{ng}{k}.$$

Since E is the point of intersection of the entering ray with the axis, $HE = f$; and likewise E', being the point of intersection of the emergent ray with the axis, $H'E'$ is equal to nf_3. Equation (5) may be written,

$$f \cdot nf_3 - \left(-\frac{ng}{k}\right)f - \frac{l}{k} \cdot nf_3 = \frac{nh}{k},$$

or,

$$\left(f - \frac{l}{k}\right)\left(nf_3 - \frac{(-ng)}{k}\right) = \frac{nh}{k} - \frac{nlg}{k^2} = -\frac{n}{k^2}(gl - hk).$$

But by a property of continuous fractions, $gl - hk = 1.$ *

Hence, $$\left(f - \frac{l}{k}\right)\left(nf_3 - \frac{(-ng)}{k}\right) = -\frac{n}{k^2}. \qquad (6)$$

We have also (Fig. 22) $FE = HE + HF$, but HF is in the figure negative; if it were positive, we should have $FE = HE - HF$; or, $FE = f - \frac{l}{k}$.

Similarly, $$F'E' = nf_3 - \frac{(-ng)}{k}.$$

Let the distance FE be denoted by u. As in the case of single lenses, it will be most convenient to consider u positive when E lies to left of F, and negative when it lies to the right of F, as in our

* Reference to the simpler convergents on page 71 will make this apparent.

figure.* Likewise if the distance $F'E'$ be denoted by u', then u' will be positive when E' lies to the right of F', and negative when it lies to the left of F', as in the figure. Then

$f - \frac{l}{k}$ becomes $-u$, and $nf_3 - \frac{(-ng)}{k}$ becomes $-u'$;

and equation (6) thus becomes

$$uu' = -\frac{n}{k^2} = -\frac{1}{k} \cdot \frac{n}{k}.$$

This equation is true for any two conjugate points E and E'; and to find the values of u and u' which

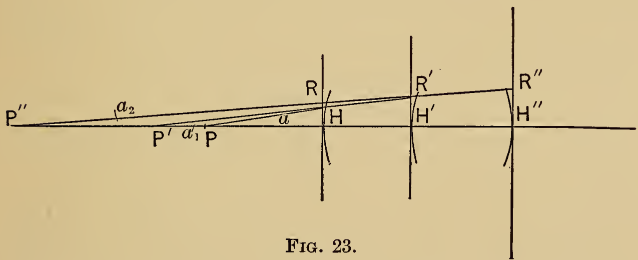

FIG. 23.

give to E and E' the properties of principal points, we make use of Helmholtz' formula for obtaining the relation between the size of an object and its image. In Fig. 23 let H, H', and H'' represent the points of intersection of the refracting surfaces with the axis; then RH, $R'H'$, and $R''H''$ are the principal planes of these surfaces. A ray of light passing

*See Chap. II., p. 35.

through P meets the principal plane RH at R; it is refracted to R', where it is again refracted to R''. RPH, the angle which the incident ray makes with the axis, is called a, and the angle $RP'H$, which the refracted ray makes with the axis, is called a_1. Similarly, a_2 and a_3 are the angles which the ray makes with the axis after the second and third refractions. Let b, b_1, b_2, represent the distances from the axis at which the ray meets the successive principal planes. As in our previous demonstrations, the refractive index of air is unity; that of the aqueous and vitreous are each n, that of the crystalline lens n'; the distance HH' is nt_1, and $H'H''$ is $n't_2$. From Fig. 23 we obtain the following equations:

$$HP = \frac{b}{\tan a}; \quad HP' = \frac{b}{\tan a_1}; \quad H'P'' = \frac{b_1}{\tan a_2}.$$

Since in the first refraction HP and HP' are conjugate foci, and both negative, we have formula (b), page 31,

$$-\frac{1}{HP} + \frac{n}{HP'} = \frac{1-n}{r}, \text{ or } -\frac{\tan a}{b} + \frac{n \,.\, \tan a_1}{b} = k_0,$$

from which $$n \cdot \tan a_1 = \tan a + k_0 b.$$

Referring again to Fig. 23, we see that

$$b_1 = b + nt_1 \cdot \tan a_1.$$

In exactly the same way it is proved that

$$n' \tan a_2 = n \cdot \tan a_1 + k_1 b_1, \text{ and } b_2 = b_1 + n' t_2 \cdot \tan a_2,$$

and so on. By these equations all the quantities $n \cdot \tan a_1$, b_1, $n' \tan a_2$, b_2, $n \cdot \tan a_3$, may be expressed in terms of $\tan a$ and b. Their values become

$$n \tan a_1 = k_0 b + \tan a,$$

$$b_1 = (k_0 t_1 + 1) b + t_1 \tan a,$$

$$n' \tan a_2 = \{k_1(k_0 t_1 + 1) + k_0\} b + (k_1 t_1 + 1) \tan a,$$

and so on.

The coefficients of b and $\tan a$ in these equations are seen to be respectively the numerators and denominators of the successive convergents on page 71.

Hence we may write out the value of b_2 thus: $b_2 = gb + h \tan a$; and $n \cdot \tan a_3 = kb + l \cdot \tan a$, where g, h, k, and l have the same significance as on page 71.

If o represent the linear dimension of an object and i_1, i_2, and i_3 the corresponding dimensions of the successive images, then from Helmholtz' formula, page 39, we shall have

$$o \cdot \tan a = n \cdot i_1 \cdot \tan a_1 = n' \cdot i_2 \cdot \tan a_2 = n \cdot i_3 \cdot \tan a_3.$$

The relation between the object and its final image is

$$o \cdot \tan a = i_3 \cdot n \cdot \tan a_3. \tag{7}$$

But $n \cdot \tan a_3 = kb + l \cdot \tan a$, and from the figure, $b = HP \tan a$, or since $HP = -f$, $b = -f \cdot \tan a$.

Substituting this value, we have

$$n \cdot \tan a_3 = l \cdot \tan a - kf \cdot \tan a,$$

or
$$n \cdot \tan a_3 = k \cdot \tan a \left(\frac{l}{k} - f\right),$$

and equation (7) becomes

$$o \cdot \tan a = i_3 k \cdot \tan a \left(\frac{l}{k} - f\right).$$

Referring to page 75, we see that

$$\left(\frac{l}{k} - f\right) = u.$$

Therefore $o = i_3 ku$, and if object and image are equal, which is the condition to be imposed in order that u and u' may determine the principal points,

$$o = i_3 \quad \text{and} \quad u = \frac{1}{k}.$$

From the equation

$$uu' = -\frac{n}{k^2}, \text{ we find } u' = -\frac{n}{k}.$$

Thus we have determined the values of u and u' which represent the distances of the first and second principal points from the first and second principal foci, respectively; and since these distances are the

first and second principal focal distances, we have only to find the numerical value of k in order that the solution of the system be complete. The value of k can be obtained if we know the radii of the refracting surfaces, the distances between these surfaces, and the refractive indices of the media. In the human eye these quantities have all been meas-

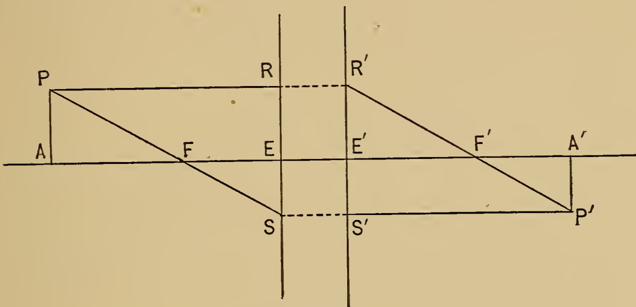

FIG. 24.

ured by careful and scientific investigators. Of these the name of Helmholtz is most conspicuous.

Before substituting these values it will be well to show the construction of the image after refraction through a number of media, and to demonstrate the properties of the nodal points. The geometrical construction of the image is almost a repetition of that for single lenses.

In Fig. 24, RS and $R'S'$ represent the principal planes, F and F' the principal foci, AP the linear

dimension of the object, and $A'P'$ that of the image. The anterior focal distance EF is denoted by F, and the posterior focal distance $E'F'$ by F'. If, as in our former notation, u denote the distance of the point A from the anterior focus, and u' the distance of its image from the posterior focus, then $FA = u$ and $F'A' = u'$. From the similar triangles PAF and EFS we have

$$\frac{PA}{ES} = \frac{FA}{EF},$$

and as $ES = A'P'$, $$\frac{PA}{P'A'} = \frac{FA}{EF}.$$

Since in our construction the object and image are on opposite sides of the axis, the image i must be considered negative, and we have this equation to determine the size of the image :

$$\frac{o}{-i} = \frac{u}{F}, \text{ or } i = -o\frac{F}{u}.$$

Also from the triangles $E'F'R'$ and $F'A'P'$, we obtain the relation

$$\frac{o}{-i} = \frac{F'}{u'}, \text{ or } i = -o\frac{u'}{F'}.$$

This construction differs from that for single lenses only in the respect that in lenses F and F' are equal, while in the present case F is to F' as

1 to n, that is, as the index of air is to that of the final medium or vitreous. If, after refraction through any number of media, light enters a medium of the same index as that of the first medium, the system acts toward light as a single lens; if the final medium has not the same index as the first, the system is analogous to a single refracting surface.

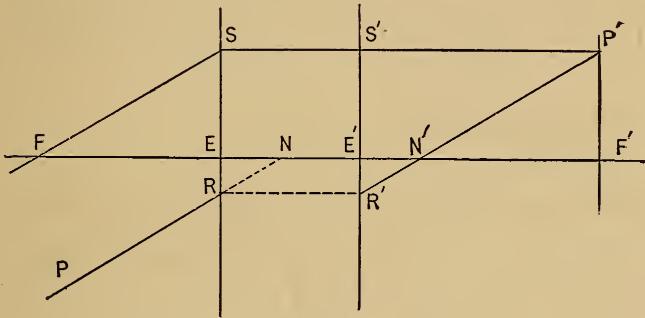

Fig. 25.

We have seen that in refraction at one surface there is one nodal point at the centre of curvature, and in refraction through lenses there are two nodal points. The characteristic property of nodal points, we remember, is that a ray directed toward one of them appears after refraction to come from the other, and in a direction parallel to that before refraction. Let us then take a ray PR (Fig. 25) directed toward the point N on the axis, so that after refraction through the system it appears to pass through

N' in a direction $N'P'$ parallel to PR; then N and N' are nodal points. To find the position of these points we construct the principal planes RS and $R'S'$ and the principal focal plane $F'P'$. The ray PR meets the first principal plane at R. It must after refraction meet the second principal plane at R' so that $ER = E'R'$. P' is its point of intersection with the focal plane $P'F'$. If we draw any two rays from P', passing through the system, they

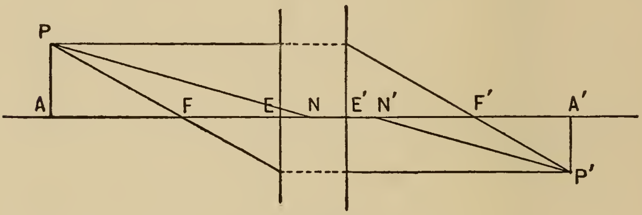

FIG. 26.

must after refraction be parallel. Let $P'N'$ represent one of these rays, and $P'S'$, parallel to the axis, the other. $P'S'$ must after refraction pass through the anterior focus F. SF will represent this ray and RP will represent the ray $P'N'$ after refraction. FN, the distance from the first focus to the first nodal point, is $FE + EN$. The triangle REN is equal to $R'E'N'$, and FES is equal to $N'F'P'$; hence $FE + EN = E'N' + N'F'$, or $FN = F'$. In the same way we find $N'F' = F$. When, as in lenses,

F and F' are equal, it is clear that the nodal and principal points coincide. It is also apparent that the distance between nodal points is the same as that between principal points. The size of the image may be determined if we know the distance of object and image from the first and second nodal points respectively. In Fig. 26 $AP = o$, and $A'P' = i$; then from the similar triangles APN and $A'P'N'$,

$$\frac{AP}{A'P'} = \frac{AN}{A'N'}, \text{ or } \frac{o}{-i} = \frac{AN}{A'N'}.$$

The following is a table of measurements of the constants which we shall require in the determination of the refractive power of the eye:

Radius of curvature of the cornea (r) .	7.829 mm.
Radius of curvature of the anterior surface of the lens in a state of rest (r')	10 mm.
Radius of curvature of the posterior face of the lens (r'')	6 mm.
Distance of the anterior surface of the cornea from the anterior pole of the lens (nt_1)	3.6 mm.
Thickness of the lens ($n't_2$)	3.6 mm.
Index of refraction of the cornea, aqueous and vitreous (n)	1.3365 mm.
Equivalent index of refraction of the crystalline lens (n')	1.4371 mm.

Substituting these values in the successive convergents on page 71, we deduce the following :

$$k_0 = \frac{1-n}{r} = -.043$$

$$t_1 = \frac{3.6}{n} = 2.6936$$

$$t_2 = \frac{3.6}{n'} = 2.505$$

$$k_1 = \frac{n-n'}{r'} = -.01$$

$$k_2 = \frac{n'-n}{-r''} = -.0168$$

$$k_0 t_1 + 1 = .8842, \qquad k_1 t_1 + 1 = .9731,$$

$$k_1(k_0 t_1 + 1) + k_0 = -.0518,$$

$$t_2\{k_1(k_0 t_1 + 1) + k_0\} + k_0 t_1 + 1 = g = .7544,$$

$$t_2(k_1 t_1 + 1) + t_1 = h = 5.1312,$$

$$k_2[t_2\{k_1(k_0 t_1 + 1) + k_0\} + k_0 t_1 + 1]$$
$$+ k_1(k_0 t_1 + 1) + k_0 = k = -.0645,$$

$$k_2\{t_2(k_1 t_1 + 1) + t_1\} + k_1 t_1 + 1 = l = .8869.$$

The first principal focal distance EF, which is equal to $\frac{1}{k}$, thus becomes

$$-15.5038 \text{ mm.}; \text{ and } HF \text{ or } \frac{l}{k} = -13.7504 \text{ mm.}$$

Therefore EH, which is equal to $EF - HF$, is -1.7534 mm.; thus the cornea lies 1.7534 mm. in front of the first principal point. The second focal distance $E'F'$, being represented by $-\frac{n}{k}$, is

$$20.721 \text{ mm.}; \text{ and } H''F' \text{ or } -\frac{ng}{k} \text{ is } 15.6326 \text{ mm.}$$

Therefore E', the second principal point, lies 5.0884 mm. in front of the posterior surface of the lens, and since this surface is 7.2 mm. behind the anterior surface of the cornea, the second principal point lies 2.1116 mm. behind the cornea. The distance between the two principal points is .3582 mm. The position of the first nodal point of the eye is, as we know, found by laying off the distance FN equal to $E'F'$, the second focal distance; as thus found, N lies 6.9706 mm. behind the anterior surface of the cornea; and since the distance between nodal points is the same as that between principal points, the second nodal point is 7.3288 mm. behind the anterior surface of the cornea. With these points determined our knowledge of the eye as an optical system is

complete. The following table giving the positions of these points will be found useful for reference :

Distance of the anterior surface of the cornea from the first principal point	1.7534 mm.
Distance of the anterior surface of the cornea from the second principal point	2.1116 mm.
Distance of the anterior surface of the cornea from the first nodal point .	6.9706 mm.
Distance of the anterior surface of the cornea from the second nodal point	7.3288 mm.
Distance of the first principal focus from the anterior surface of the cornea	13.7504 mm.
Distance of the second principal focus from the anterior surface of the cornea	22.8326 mm.

The hypothetical eye which possesses these cardinal points is called the **schematic** eye.

We have seen that the method of determining the size and position of the image is the same in refraction at one surface as in refraction at any number of surfaces, with the exception that there are two principal and two nodal points in the latter case and only one principal point in the former, with one nodal point at the centre of curvature. We also know that the size of the image is proportional to the

anterior focal distance; and that this distance is measured from the first principal point, while the second focal distance, which determines the position of the image, is measured from the second principal point. In the eye the two principal points are so near to each other that they may with very slight error be merged into one. In this way the optical effect of the eye may be represented by one refracting surface whose summit coincides with the merged principal points. Listing first proposed this simplification, and called the resulting equivalent the **reduced eye.*** In this reduction the single principal point is between the two principal points of the schematic eye as determined by Listing.† The ratio of F to F' remains unchanged, and as F is to F' as 1 to n, we can determine what value n must have in the reduced eye. Listing takes as the anterior focal distance 15.036 mm., and as the second focal distance 20.133 mm., from which we find n is equal to 1.3365, and this, as a reference to the table on page 83 will show, is the index of the vitreous. From the equation $F = \frac{r}{n-1}$, we can determine the

* *Dioptrik des Auges*, Wagner's "Handwörterbuch der Physiologie."

† Listing's schematic eye differs slightly from that given in the text, since he takes the radius of curvature of the cornea as 8 mm., and the distance from cornea to lens and the thickness of the lens as each equal to 4 mm.

radius of curvature of the reduced eye. Using Listing's measurements, this is 5.1248 mm. Figure 27 illustrates in (1) the *schematic* eye and in (2) the *reduced* eye.

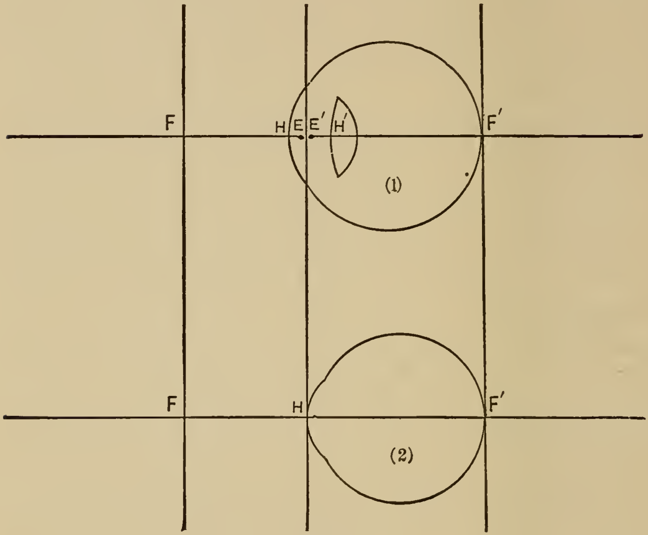

FIG. 27.

Donders has furnished a less accurate but a more useful reduced eye than that of Listing. He neglects fractions, making the anterior focal distance 15 mm., the posterior focal distance 20 mm., and the radius of curvature 5 mm. From $F = \dfrac{r}{n-1}$ we find $n = 1.333$, which is the index of water. Calculations

as to the size of images with this eye are extremely simple. If it be remembered that the size of the image is proportional to the anterior focal distance, and that in Donders' eye this is 15 mm., while in the schematic eye it is 15.5038 mm., it will be apparent that the image obtained from calculation with Donders' eye is to the actual image in a normal eye as 15 to 15.5038.

Based upon these data, artificial eyes have been constructed for the study of the refraction of the eye.

CHAPTER V

THE DETERMINATION OF THE CARDINAL POINTS OF THE EYE IN COMBINATION WITH A LENS

We must now carry our investigations one step further in order to appreciate the effect upon vision of placing a lens in front of the eye. A lens has two refracting surfaces, and the eye three such surfaces; hence, if we wish to use the method given in the preceding chapter, we must write out the convergents for two additional surfaces; but simpler than this would be an independent geometrical construction, from which we could find the effect of combining two optical systems. But for our purpose it suffices to make use of the reduced eye of one surface, since we do not care to know the exact size of the retinal image. What we wish to know is the relative size of images with lenses and without them; and the investigation with the reduced eye furnishes this information.

In Fig. 28 let A represent the centre of a spherical lens; and since we neglect the thickness of the lens, the letter A also marks the position of the merged principal points of the lens. Let A' repre-

sent the position of the principal point of the reduced eye. We know that in the lens A the proportion of curvature at the two faces is immaterial, in other words, any thin bi-spherical lens may be replaced by a plano-spherical lens of the same focal length ; and for the sake of simplicity we shall make this substitution. To find the cardinal points of the combination of eye and lens, we use the formulæ deduced in the preceding chapter. If F denote the anterior and

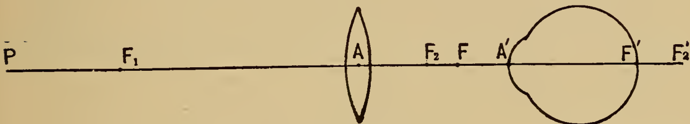

FIG. 28.

F' the posterior focal length of the combination, then $F = \frac{1}{k}$, and $F' = -\frac{n}{k}$, in which n represents the index of the reduced eye, that is, if we use Listing's reduced eye, it is equal to the index of the vitreous, and if we use Donders' reduction, n is equal to the index of water. The thickness of the lens, which in our formula is represented by nt_1, is equal to zero ; and the distance of the lens from the eye, which is represented by $n't_2$, becomes t_2, since n', being now the index of the air, is unity. This distance between the eye and lens may be conveniently denoted by the letter e ; then t_2 in the formula is replaced by e.

Thus we have $k_0 = \frac{1-n}{r}$, in which r represents the radius of curvature of the first surface of the lens and n the index of the lens; in the plano-spherical lens $\frac{r}{1-n}$ is equal to the focal length of the lens. If we denote this by F_1, we have $k_0 = \frac{1}{F_1}$. The radius of curvature of the second face of the lens is infinite, hence $k_1 = 0$. In the same way we find $k_2 = \frac{1}{F_2}$, F_2 being the anterior focal length of the eye. Making these substitutions in the expressions (g), (k), and (l), page 71, we derive

$$g = \frac{e + F_1}{F_1},$$

$$k = \frac{e + F_1 + F_2}{F_1 F_2},$$

$$l = \frac{e + F_2}{F_2}.$$

Hence,
$$F = \frac{1}{k} = \frac{F_1 F_2}{F_1 + F_2 + e};$$

$$F' = -\frac{n}{k} = -\frac{nF_1 F_2}{F_1 + F_2 + e} = -\frac{F_1 F_2'}{F_1 + F_2 + e}.$$

With the convention of signs which we are using, F_1 and F_2, being measured to the left of A and A'

respectively, are both negative. In order that the formulæ may be applicable in any case without prefixing the minus sign to F_1 and F_2, we must change the sign of these quantities. Making this alteration, the equations become

$$F = \frac{1}{k} = -\frac{F_1 F_2}{F_1 + F_2 - e}; \quad F' = -\frac{n}{k} = \frac{F_1 F_2'}{F_1 + F_2 - e}.$$

It is in this form that these equations are usually written. To find the distance from the lens to the anterior focus F, we have

$$AF = \frac{l}{k} = -\frac{F_1 (F_2 - e)}{F_1 + F_2 - e};$$

and to find the distance from the principal point A' of the eye to the posterior focus F, we have

$$A'F' = -\frac{ng}{k} = \frac{F_2' (F_1 - e)}{F_1 + F_2 - e}.$$

Having found the positions of the two foci and the two focal distances, we know also the positions of the two principal points, and the solution of the system is complete.

CHAPTER VI

ERRORS OF REFRACTION—LENSES USED AS SPECTACLES

We have studied the eye as an optical system, taking as our measurements those found to exist with close approximation to uniformity in a large number of eyes. The posterior foc~~al~~-us ~~distance~~ of this system we have found to lie 22.8326 mm. behind the anterior surface of the cornea. When the retina lies at the same distance from the cornea, the image of a distant object will be accurately formed on the retina. When the retina lies in front of the focus of the eye, the image of a distant object will be blurred. This condition of the eye is called **hyperopia**; and when the retina lies behind the focus, the resulting condition is called **myopia**.

Eyes so constituted that retina and principal focus do not coincide are said to be **ametropic**, or affected with **errors of refraction**. When retina and focus coincide, as in the normal eye, the condition is called **emmetropia**. A distant object will be clearly seen by a healthy emmetropic eye, but the image of a near object will fall behind the retina. In order

that a near object be clearly seen, either the eye must be elongated or the focus must be brought forward. The latter change is the one which occurs by an increase in curvature of the crystalline lens (principally of the anterior surface of the lens) under the influence of the ciliary muscle. This change in curvature is called **accommodation.** By it we are enabled to adapt the eye for varying distances.

Since the curvature of the lens is increased during accommodation, the optical system of the eye in this state differs from that when the eye is adapted for a distant object; the focal distance has been shortened by the exercise of accommodation, and the retina now lies behind the principal focus; in other words, the eye has become *myopic.* Thus we see that the emmetropic eye can render itself myopic in order to see near objects. Similarly, a hyperopic eye, possessing sufficient accommodative power, may become emmetropic to see distant objects, and by a still further increase in curvature of the lens may even become myopic and thus see clearly near objects.* The myopic eye is unable to increase its focal distance, thus bringing the principal focus back to the retina; it cannot therefore see distant objects clearly.

Since an eye may be myopic either from increase in curvature or from increase in the antero-posterior

* These conditions, however, are not included in the usual acceptation of the words *emmetropia* and *myopia.*

diameter, we have *curvature* myopia and *axial* myopia. Similarly, we have *curvature* hyperopia and *axial* hyperopia. In axial hyperopia and myopia the eye as an optical system is normal, the defect being in the position of the retina; in curvature ametropia the deviation from the normal is in the optical system.*

Without using the formulæ for the cardinal points of the eye and lens in combination, we can easily

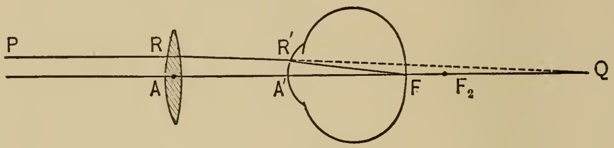

FIG. 29.

understand how lenses can in hyperopia and myopia bring the retina and principal focus into coincidence. Let $A'R'$ (Fig. 29) represent the cornea of a hyperopic eye whose principal focus is at F_2 behind the retina. Rays of light parallel to the axis before entering the eye will after refraction meet at F_2. In order to bring these rays to a focus at F on the retina, we introduce a convex lens A. Let Q be the principal focus of this lens; then a ray PR parallel to the axis will, after passing through the

* *Index* ametropia, which in optical effect resembles curvature ametropia, occurs in exceptional cases.

lens, take the direction RQ; but at R' it is refracted by the eye and assumes the direction $R'F$.

Let $A'R'$ (Fig. 30) represent the cornea of a myopic eye whose focus F_2 lies in front of the retina; then rays of light parallel to the axis before entering the eye will, after refraction, meet at F_2. A ray passing through the point Q will meet the axis at its conjugate focus *behind* the principal focus. Let Q be so taken that its conjugate lies on the retina; then Q is the **far point** of the eye, since light

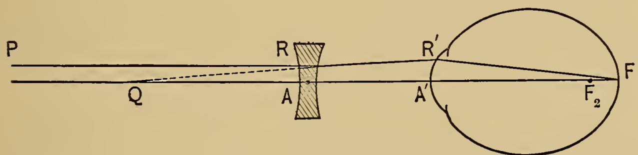

FIG. 30.

from any more distant point will meet the axis in front of the retina. If we place at A a concave lens whose focal length is AQ, then any ray parallel to the axis will, after refraction through the lens, appear to pass through Q, and its direction will be QR'. But the ray QR' will, after refraction by the eye, meet the axis at F. Hence a concave lens whose focal length is AQ will bring parallel rays to a focus on the retina.

It is clear from these diagrams that the farther from the eye the convex lens is placed the weaker

is the lens required to bring the retina and focus into coincidence, since AQ is the focal length of the required lens. In the case of myopia the opposite is true, for AQ, the focal length of the lens, diminishes as the lens is removed from the eye.

We can determine the amount of shortening of the eye in hyperopia, or of lengthening in myopia, from the equation

$$\frac{F}{f} + \frac{F'}{f'} = 1.$$

In this equation F and F' are the focal distances of the eye, and f and f' are the conjugate distances $A'Q$ and $A'F$.

This equation is true not only for refraction at one surface and for lenses, as we have seen; it applies equally to a system of refracting surfaces. That this is so, follows readily from the equation

$$uu' = -\frac{1}{k} \cdot \frac{n}{k},$$

since $u = f - F$, $u' = f' - F'$, and $-\frac{1}{k} \cdot \frac{n}{k} = FF'$.

Applying this equation in myopia, if Q, the far point of the eye, is 100 mm. from the first principal point of the eye, then f is 100 mm. Substituting for F, the anterior focal distance, its value 15.5038 mm., and for F' its value 20.721 mm., we find the corre-

sponding value of f' to be 24.5219 mm. That is, the retina lies 24.5219 mm. behind the second principal point of the eye. Subtracting 20.721 mm., which is the distance of the retina from the second principal point in the normal eye, we have 3.8009 mm. as the amount of lengthening.

In hyperopia if rays directed toward Q (Fig. 29) would be focused on the retina, and if the distance of Q from the first principal point of the eye is 100 mm., then f in our equation becomes -100 mm. Making the proper substitutions as before, we find f' to be 17.9402 mm., and the amount of shortening is 2.7808 mm. From this we see that the shortening in hyperopia is less than the lengthening in the same degree of myopia.

Since the strength of the lens required to correct an error of refraction varies with the position of the lens, it is clear that we cannot measure the error by the correcting lens unless we adopt some fixed position at which the lens should be placed. To be strictly correct, this should be at the first principal point, from which Q is measured, but as this is impossible, the position at which spectacle lenses are ordinarily worn, about 15 mm., in front of the cornea, is taken as the standard. In our example of the myopic eye, 100 mm., as measured from the first principal point E (Fig. 22) indicates the position of the far point in a myopia of 10 dioptres; but the

focal length of the correcting lens, placed 15 mm. in front of the cornea and consequently nearly 17 mm. in front of E, must be a trifle more than 83 mm. Taking this as 83 mm., the dioptric power of the lens is slightly more than 12 D. In the second example we have a hyperopia of 10 dioptres as measured from E, while the degree, as indicated by the correcting glass placed 15 mm. in front of the cornea, is about 8.5 D.

If the crystalline lens has been extracted from the eye, there is only one refracting surface and one medium, for the index of the aqueous and vitreous are identical. The second principal focal distance of the aphakic eye is therefore derived from the equation

$$F' = \frac{nr}{n-1},$$

and from this the focus is found to lie 31.095 mm. behind the anterior surface of the cornea. The dioptric power of the aphakic eye is therefore 32.16 D. The principal focus of the normal eye lies 22.8326 mm. behind the anterior surface of the cornea; hence its dioptric power is 43.8 D. Subtracting the power of the aphakic from that of the normal eye, we find that the action of the crystalline lens, when adapted for distant vision, is nearly equivalent to that of a lens of 11.5 D. placed in contact with the cornea.

After extraction of the lens from an emmetropic eye, a convex lens of 11.5 D. would be required to bring the image of a distant object to an accurate focus on the retina, if the lens were worn in contact with the cornea; but as this is impracticable, a weaker lens is required. If the distance of the lens from the cornea be 15 mm., then its focal length must be 87 mm. (the focal length of a lens of 11.5 D.) + 15 mm. A lens having this focal length represents a dioptric power slightly in excess of 10 D. Hence after extraction of the lens from an emmetropic eye, we should expect a lens of 10 D. to rectify the refractive condition for distant objects, and this corresponds very closely with what we find in practice.

Let us now examine the condition which exists after extraction of the lens from a myopic eye. If the myopia is caused by excessive curvature of the cornea, the amount of myopia relieved by the extraction is the same as in emmetropia, that is, if there existed prior to extraction 11.5 D. of myopia as measured from the cornea, the eye would be rendered emmetropic by the extraction. This, however, is not so when, as is usually the case, the myopia is due to lengthening of the antero-posterior diameter of the eye. As previously shown, the posterior focus of the aphakic eye of normal curvature lies 31.095 mm. behind the cornea. Hence if an eye

is emmetropic after extraction of the lens, the retina also must lie at this distance from the cornea. To find what degree of myopia exists in an eye whose retina lies at this distance from the cornea, we must use the equation

$$\frac{F}{f}+\frac{F'}{f'}=1.$$

As we can find from this equation the amount of lengthening, if we know the distance (f) of the far point from the eye; so if we know the distance (f') of the retina from the second principal point, we can find the distance (f) of the far point from the eye, and this distance measures the myopia. Our result will be sufficiently accurate if we consider F as $15\frac{1}{2}$ mm., F' as 21 mm., and f' as 29 mm. We find this last value by subtracting 2.1116 mm., the distance of the second principal point from the cornea, from 31.095 mm., the distance of the retina from the cornea, the result being the distance (f') of the retina from the second principal point. Making these substitutions, we find the corresponding value of f to be 56.2 mm. This is the distance of the far point of the eye from the first principal point; its distance from the cornea is therefore approximately 54.4 mm.; or, expressed in dioptres, the myopia, as measured from the cornea, is 18.3 D.

To find what is the required power of the correcting

lens for this amount of myopia we shall consider the lens placed not, as before, 15 mm. from the cornea, but 15 mm. from the first principal point, or about 13 mm. from the cornea. The position at which spectacle glasses are usually worn is more remote from the eye than as indicated in the standard (15 mm.) which we have taken, but this is not so when very strong concave lenses are worn. These lenses, when tolerated, are worn very near the eye; hence we make the change in distance so as to be more in accord with the actual conditions with which we meet. It will be noticed that a slight change in position of these strong lenses makes perceptible change in dioptric power. The focal length of the correcting lens in the case which we are considering thus becomes 56.2 mm. – 15 mm., or 41.2 mm. The power of the lens is accordingly 24 D. From this we see that the eye must have a myopia of 24 D. as measured by its correcting lens in order to become emmetropic after extraction of the lens from the eye.

Let us now deduce from calculation the lens required for distant vision in an aphakic eye, in which, before extraction, a concave spherical lens of 20 D. was found to correct the ametropia. Making the proper substitutions, we first find that in this case the retina lies 29.7 mm. behind the cornea, and since the posterior focus of the aphakic eye is 31 mm. behind the cornea, the eye will be hyperopic.

In the equation $\frac{F}{f} + \frac{F'}{f'} = 1$, F now represents the anterior focal length of the aphakic eye; it is therefore approximately 23 mm.; F' represents the posterior focal length, approximately 31 mm.; f' represents the distance $A'F$ (Fig. 29), and in this case is 29.7 mm. From these data we can find f ($A'Q$, Fig. 29), the focal length of the required lens if placed in contact with the cornea. We thus derive the value $f = -512$ mm. Supposing that the lens will be worn in the same position as the concave lens previous to extraction, we add 13 mm. to 512 mm. The dioptric power of the lens required to correct the existing hyperopia is therefore 1.9 D. This result agrees very closely with the condition as found to exist in a particular case after the removal of the transparent lens for the cure of myopia. So close an approximation could not be obtained in every case, however, for in high myopia there is frequently defect in curvature and in position of the lens as well as in length of the eye.

From the foregoing studies we also learn the effect of a change in the position of the crystalline lens upon the refractive power of the eye; if the lens move toward the cornea the eye will be rendered myopic, and if it move away from the cornea the eye will become hyperopic.

We shall next study the effect of changing the

position of spectacle lenses when used for near vision. We have seen that convex lenses used to aid the distant vision of hyperopes increase in power as they are withdrawn from the eye. When convex lenses are worn by presbyopes to replace the accommodation which has failed, the effect of withdrawing the lens varies under different circumstances. Suppose first that we are dealing with an emmetropic eye whose accommodative power has entirely failed; this eye can focus only parallel rays upon the retina.

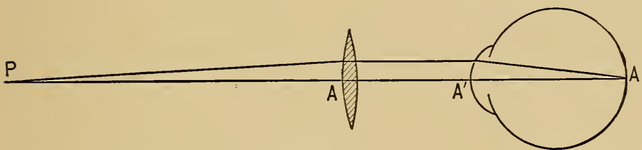

Fig. 31.

Let P (Fig. 31) represent the point where a near object intersects the axis; then, in order that the object be clearly seen by the eye without accommodation, a convex lens A, whose focal length is AP, must be placed before the eye. As the lens is removed from the eye the focal length AP diminishes, and the strength of the lens necessary to produce distinct vision must be increased.

We shall next take the case of a hyperopic eye whose accommodative power has been lost. Let P (Fig. 32) represent the intersection of a near object with the axis; then the conjugate of the point P in

the refraction by the lens is Q, which lies behind the retina. If the position of Q be such that it is conjugate to R in the refraction by the eye, then the object at P will be accurately focused on the retina

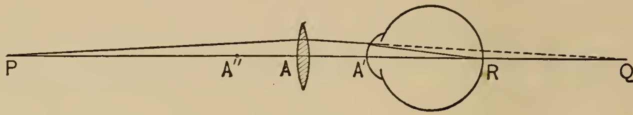

FIG. 32.

with the aid of the lens. If P remain stationary while the lens is moved to A'', the effect upon the power of the lens will not be the same in all cases. We have the equation

$$\frac{1}{AP}+\frac{1}{AQ}=\frac{1}{F}, \quad \text{or} \quad \frac{1}{f}+\frac{1}{f'}=\frac{1}{F}.$$

When the lens is moved from A to A'', the distance f is diminished by AA'', and f' is increased by AA''. To study the effect of this change upon the equation, we take the case when f and f' are equal, each being then equal to $2F$. The equation then becomes

$$\frac{1}{2F}+\frac{1}{2F}=\frac{1}{F}.$$

When the lens is at A'' we have the similar equation

$$\frac{1}{2F-AA''}+\frac{1}{2F+AA''}=\frac{1}{F_1},$$

from which

$$\frac{1}{F_1} = \frac{4F}{4F^2 - (AA'')^2} = \frac{1}{F - \frac{(AA'')^2}{4F}}.$$

From this we have $F_1 = F - \frac{(AA'')^2}{4F}$. Therefore, F_1 is less than F, and the dioptric power of the lens must be increased in order that Q may be conjugate to P. In other words, for a fixed lens the line PQ is shorter when AP and AQ are equal than for any other position of the lens; conversely, if the line PQ remains constant, a stronger lens is required when it is moved from its central position at A, whether it is moved toward P or toward Q. Hence we have this rule: If the distance between object and lens is less than twice the focal length of the lens, the power of the lens is diminished by moving it from the eye and toward the object; if, on the other hand, the distance between object and lens is more than twice the focal length of the lens, the power of the latter is increased by moving it away from the eye. It will be observed that the distance at which a book is held for reading is ordinarily less than twice the focal length of the lens worn as a reading glass; consequently the power of such a glass is weakened by moving it from the eye.

In Fig. 33 we have a representation of an eye myopic either from abnormal length of axis or from

act of accommodation. Without the aid of a lens the conjugate of Q is at R on the retina; but in order that an object at P, nearer than Q, may be seen, a convex lens is required; then in the refraction by the lens, P and Q are conjugate. It is seen that AP is positive and AQ is negative; therefore AP must be less than $2F$, for it must be less than F in order that AQ may be negative. Hence, as in the

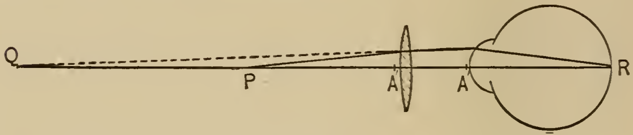

FIG. 33.

previous case, the power of the lens will be diminished by moving it away from the eye.

From this we see that the statement that presbyopes increase the power of their glasses by moving them to the tip of the nose is inaccurate. The power of the lens can be increased in this way only on condition that the object is moved in the same direction and to the same extent as the lens; but as the object is removed from the eye, the retinal image becomes smaller — a manifest disadvantage.

CHAPTER VII

THE EFFECT OF SPHERICAL LENSES UPON THE SIZE OF RETINAL IMAGES

Spherical lenses used for the purpose of bringing the image of an object to an accurate focus on the retina have also an effect upon the size of retinal images. In many cases this is so slight as to escape notice; but when strong glasses are worn the effect is considerable. If the glass before one eye differs much in strength from that before the other eye, confusion results from the unequal images, and such glasses are usually rejected. The size of the image after refraction through any optical system is obtained, as we have learned, from the equation

$$i = -o \cdot \frac{F_2}{u},$$

where u is the distance of the object from the anterior focus of the system, and F_2 is the anterior focal distance. This equation gives a negative value for the image when it is real, since such an image lies on the opposite side of the axis to the object, and is consequently inverted. As we shall have in

our further studies to deal only with retinal images, we shall neglect the minus sign, because the image is seen by the observer as erect, the transformation being made by mental act. Hence we shall consider as positive those images which appear to be erect; and we shall consider as negative those which appear to be inverted. Our equation thus becomes

$$i = o \cdot \frac{F_2}{u}.$$

If now a lens be introduced before the eye, we shall have a new optical system; and, as before, the size of the image formed by this system will be found from the equation

$$i' = o \cdot \frac{F}{u_1},$$

where u_1 represents the distance of the object from the new anterior focus, and F represents the new focal distance. Hence the relation of the size of the image with the lens to that without the lens is expressed by the equation

$$\frac{i'}{i} = \frac{u \times F}{u_1 \times F_2}.$$

When the distance of the object is great in comparison with the change in position of the anterior

focus caused by adding the lens, then u and u_1 may be considered identical. As this is true in the case of lenses used as spectacles, we have the simplified expression

$$\frac{i'}{i} = \frac{F}{F_2}.$$

But on page 93 we learned that the anterior focal distance of the eye combined with a lens is derived from the equation

$$F = \frac{F_1 F_2}{F_1 + F_2 - e}.$$

Making this substitution, we have

$$\frac{i'}{i} = \frac{F_1}{F_1 + F_2 - e}.$$

In this equation F_1 is the focal length of the lens, F_2 is the anterior focal length of the eye, and e is the distance between eye and lens. This formula is sufficiently accurate to determine the effect of spectacles upon retinal images except when the object is very near the eye; but as we wish to show the magnifying power of lenses in all positions in front of the eye, we shall examine the condition when u and u_1 cannot be considered identical. Referring to Fig. 28 we see that $u = PF_2$, and $u_1 = PF$.

Hence $u_1 = u + FF_2 = u + (AF - AF_2)$;

but on page 93 we found that AF is equal to

$$-\frac{F_1(F_2 - e)}{F_1 + F_2 - e}.$$

We see also that

$$AF_2 = e - F_2.$$

From this we find

$$u_1 = u + \frac{(F_2 - e)^2}{F_1 + F_2 - e};$$

or,

$$\frac{uF_1 + u(F_2 - e) + (F_2 - e)^2}{F_1 + F_2 - e};$$

and

$$\frac{i'}{i} = \frac{uF_1}{uF_1 + u(F_2 - e) + (F_2 - e)^2};$$

or,

$$\frac{i'}{i} = \frac{F_1}{F_1 + (F_2 - e) + \frac{(F_2 - e)^2}{u}}.$$

This is the general expression for the magnifying power of any lens in combination with the eye, or with any other optical system. Examining this expression, we see that if $F_2 = e$, that is, if the lens be placed at the anterior focus of the eye,

$$\frac{i'}{i} = 1.$$

Hence a lens so placed has no effect upon the size of retinal images. This is true whether the lens is convex or concave, and irrespective of the distance of the object.

If the lens be convex and F_2 be less than e, that is, if the lens be without the anterior focus of the eye, $F_2 - e$ will be negative, but $(F_2 - e)^2$ will be positive. The least value which u can have is equal to $e - F_2$, since the object cannot lie between the lens and eye. When $u = e - F_2$, i' and i are equal, which indicates that there is no effect on the size of the image when object and lens are in contact. But as the distance of the object is increased and u assumes a greater value, the lens exerts a magnifying power upon the image, and this magnification increases as the distance between lens and object increases. We observe also from a further study of the equation that as e, the distance between eye and lens, varies, the magnifying power varies. When this distance becomes such that

$$(e - F_2) = F_1 + \frac{(F_2 - e)^2}{u},$$

the denominator of the expression which denotes the magnifying power becomes zero, and $\frac{i'}{i}$ becomes *infinite*.

If $(e - F_2)$ is greater than $F_1 + \frac{(F_2 - e)^2}{u}$, then $\frac{i'}{i}$ becomes *negative*.

To render the meaning of this clear, we shall first suppose the object to be so distant in comparison with the distance between lens and eye that the fraction $\frac{(F_2 - e)^2}{u}$ may be neglected as inappreciable. In this case $\frac{i'}{i}$ becomes $\frac{F_1}{F_1 + F_2 - e}$, and this expression is negative when $e - F_2$ is greater than F_1, that is, the image is inverted when the distance of the lens from the anterior focus of the eye is greater than the focal length of the lens. An aerial image of the object will then be formed by the lens in front of the eye, and a second image will be formed by the eye.*

As the object approaches the lens, the term $\frac{(F_2 - e)^2}{u}$ can no longer be ignored, and $e - F_2$, the distance of the lens from the anterior focus of the eye, must be greater than $F_1 + \frac{(F_2 - e)^2}{u}$ in order that the inverted image be formed.

The meaning of this is that as the object approaches the lens, its image recedes behind the posterior principal focus, F_1, and in order that the aerial image be formed in front of the anterior focus of the eye, the distance of the lens from the eye

* When the aerial image is near the anterior focus, the final image will lie far behind the retina, and no distinct image will be perceived by the eye.

must be greater than when the object is more remote. If we solve the equation

$$e - F_2 = F_1 + \frac{(F_2 - e)^2}{u},$$

we obtain the relation between $e - F_2$ and u, which exists when $\frac{i'}{i}$ is infinite. This relation thus becomes

$$e - F_2 = \frac{u}{2}\left(1 \pm \sqrt{1 - \frac{4\,F_1}{u}}\right).$$

Since we cannot extract the square root of a negative quantity, the fraction $\frac{4\,F_1}{u}$ must be less than 1, or equal to it, in order to give a real value to the expression. Hence we see that the least value which u can have and satisfy the condition $\frac{i'}{i} = \infty$ is $u = 4\,F_1$; and when $u = 4\,F_1$ we find the corresponding value for $e - F_2$ to be $\frac{u}{2}$. In other words, the image cannot become infinite, and consequently cannot become negative, when the distance of the object from the anterior focus of the eye is less than four times the focal length of the lens. When this distance is equal to four times the focal length of the lens, the image becomes infinite provided the lens is placed midway between the object and anterior focus of the eye. All this follows from what has been said in the preceding chapter in regard to the effect of

changing the position of a lens in near vision, and we need not have deduced this result algebraically. We have seen that PQ (Fig. 32) is shorter when the lens, A, is half way between P and Q than in any other position. Similarly, in Fig. 28, the least distance from F_2 at which P may be situated and still form an image at this point is four times the focal length of the lens, and the lens must be at the midway point of the line PF_2. Whether the lens be moved toward the object or toward the eye, the image will be thrown to the right of F_2, and the final image will be reduced in size. Thus the image attains its greatest size when the lens is in this midway position, and we have this rule for determining the effect of changing the position of the lens: As the convex lens is removed from the eye, the magnifying power increases so long as the distance of the object from the lens is *more* than twice the focal length of the lens, and when the distance between object and lens is *less* than twice the focal length of the lens, the magnifying power is diminished by further removal of the lens from the eye.

If, while the lens is convex, $F_2 - e$ be positive, that is, if the lens be placed within the anterior focus of the eye, then i' will always be less than i.

We next suppose the lens to be concave. In this case F_1 is negative. Our equation shows that when F_2 is less than e, or when the lens is without the an-

terior focus, i' is less than i, except when $u = e - F_2$; in this case, as with convex lenses, i' and i are equal. When F_2 is greater than e, i' is greater than i.

To summarize: A convex or concave lens placed at the anterior focus of the eye or of any optical system, though it alters the *position* of the image of an object, has no effect upon the *size* of the image.

A *convex* lens placed *without* the anterior focus of an optical system *magnifies* the image, the degree of magnification varying with the distance between the lens and anterior focus, and with the distance between the object and the lens.

A *convex* lens placed *within* the anterior focus of an optical system *minifies* the image, the degree of minification varying with the distance between the lens and anterior focus and with the distance between the object and lens.

A *concave* lens placed *within* the anterior focus *magnifies* the image; if placed *without* the anterior focus it *minifies* the image, the degree of magnification or minification varying with the distance between lens and anterior focus and with that between object and lens.*

* The student should verify these phenomena by taking a convex lens of 20 dioptres, which may represent the eye; if another lens be held before this and be moved to and fro while an object is viewed through the combination, the effect of the second lens upon the size of the image can be easily noted.

The anterior focus of the eye we have found to be 13.7504 mm. in front of the cornea. Spectacle glasses, being usually worn farther from the eye than this on account of the projecting eyelashes, must affect the size of retinal images; convex lenses magnify and concave lenses minify these images. Thus in the hyperopic eye, which sees with the aid of a convex lens, the image is larger than it would be in the emmetropic eye, because the optical system is the same in the two eyes, and the addition of a convex lens placed without the anterior focus magnifies the image. Similarly, in a myopic eye, which sees with the aid of a concave lens placed without the anterior focus, images will be smaller than in an emmetropic eye.

We must now compare the image as formed by the unaided hyperopic eye with that formed by the emmetropic eye. The hyperopic eye is enabled to see objects by an increase of curvature of the crystalline lens, by which means its dioptric power is increased and its focal distances are diminished. Hence the size of the retinal image is diminished by the change, since this is proportional to the anterior focal distance. The hyperopic eye without a correcting lens has smaller images than the emmetropic eye; and as the same eye with a lens has larger images than the emmetropic eye, it is evident that in hyperopia images are increased in size by the

correcting lens. The same holds true in near vision, since a greater amount of increase in curvature is necessary for the hyperopic eye to see near objects than for the emmetropic eye to see the same objects, and in consequence the image in the hyperopic eye is smaller than in the normal eye; but with the aid of the correcting lens the image is larger in the hyperopic than in the normal eye. Likewise, presbyopes, who wear convex lenses to enable them to see near objects, have larger images than those who see near objects by act of accommodation.

The myopic eye cannot adapt itself to distant vision, and hence without a lens all images of distant objects are blurred. Such an eye, however, sees near objects either without any increase of curvature or with less increase than would be required in an emmetropic eye. Thus we see that the myopic eye has the advantage of larger images of near objects than the normal eye has. If the myopia is corrected by a concave lens, the eye must now make the same effort of accommodation that the normal eye makes. The image is accordingly minified by the increase of curvature and by the concave lens if this is worn without the anterior focus of the eye. In the high degrees of myopia, in which strong glasses are required, the minifying effect is a serious obstacle to their use. In these cases the lenses should be worn as near the eyes as possible.

We shall now investigate the aphakic eye. The second principal focal distance of this eye is 31.095 mm. As the retina is only 22.8326 mm. behind the cornea, we have in the aphakic eye a high degree of curvature hyperopia. The anterior focal distance of the aphakic eye is 23.2659 mm. Hence the eye which has been deprived of its crystalline lens has larger images than the normal eye. The image, however, will be formed far behind the retina; and in order to bring it forward to the retina a strong convex lens is required. This lens worn as a spectacle glass will be within the anterior focus of the aphakic eye. It will, accordingly, reduce the size of images so that they more nearly correspond in size with those as formed in the normal eye. If the eye be hyperopic prior to the extraction of the lens, the images will be still more nearly approximated in size to those formed in the normal eye, since a stronger convex lens will be required to focus rays on the retina than if the eye had been normal. But even in high degrees of hyperopia the image will not be reduced by the correcting lens to the size of the normal image.

In the myopic eye images will be larger than if the eye had been emmetropic before extraction, because a less convergent lens will be required than if the eye had been normal. Thus we see that in all aphakic eyes the retinal images, as formed with

the aid of correcting lenses, are larger than in normal eyes, and that of aphakic eyes, those which were hyperopic prior to the extraction are least favorably situated as regards the size of images, and those which were myopic prior to the extraction are most favorably situated in this respect. To this enlargement of images is due the remarkable increase of visual acuity after extraction of the lens for the cure or improvement of myopia. We have seen that the excessively myopic eye which requires a strong concave lens to produce clear images is very much hampered by the minifying effect of the correcting lens. If, after the extraction of the crystalline lens, parallel rays of light are focused on the retina without the aid of a lens, the size of the image of a distant object, as formed in this eye, will be to the size of the corresponding image in a normal eye as the anterior focal distance of the aphakic eye is to that of the normal eye. Omitting fractions this ratio is as 23 to 15.* When we consider that, before the extraction, images were reduced in size, we are prepared to expect great improvement in visual acuity in those cases in which the operation has been successfully accomplished.

To the enlargement of images, though in a less

* This ratio refers to the linear dimensions of the images. The relative amount of retinal surface covered by the images is proportional to the squares of these numbers.

degree, is also due the excellent visual acuity which sometimes follows cataract extraction in emmetropic eyes. In the most successful cases a visual acuity of $\frac{20}{15}$ may be obtained in spite of the fact that the pupil is not entirely free from particles of opaque matter.

Finally, we shall consider the effect upon retinal images of changing the position of the convex lens in near vision. We have seen, as we might infer, that the size of the image varies according to the same rule as does the dioptric power of the lens in any position, that is, if the distance between the object and lens is less than twice the focal length of the lens, the power of the lens is weakened and the size of the image is diminished by removal of the lens from the eye; and if the distance of the object is more than twice the focal length of the lens, the power of the lens and the size of the image are increased by removal of the lens from the eye. Thus, as the prevalent belief that presbyopes increase the power of their glasses by placing them on the tip of the nose was shown to be not generally true, so the statement made in some text-books that by so placing them larger images are obtained, is also erroneous.

The condition when the eye is emmetropic and without accommodative power is shown in Fig. 31.

When one looks through an ordinary hand magni-

fying glass, the same conditions are present. If, as is usually the case, the distance of the object from the anterior focus of the eye is less than four times the focal length of the lens, the image is never inverted. If this distance is less than twice the focal length of the lens, then as the lens is removed from the eye the image is diminished in size. But when the lens A is farther from the object P than the focal length of the lens, the pencils of light after passing through the lens are convergent, and cannot be focused by a normal eye; hence the nearest point to the eye at which the lens can be placed and afford distinct vision is such that the object is at the focus of the lens, and this is for the emmetropic eye the position of greatest magnifying power. The hyperopic eye can focus the convergent pencils and receive a clear image when the lens is to the right of A; on the other hand, the myopic eye can only focus divergent pencils such as are formed when the lens is on the left of A and nearer to the object.

CHAPTER VIII

CYLINDRICAL LENSES

We have assumed that the refracting surfaces of the eye are spherical in form. This is permissible in normal cases, but in a large proportion of eyes the refracting surfaces — more especially the cornea — do not have this regularity of curvature; the curvature is found to vary appreciably in different meridians, the meridians of greatest and least curvature being usually at right angles to each other. These are called the **principal** meridians. Such a surface is called a **torus** or **toric** surface. The outer convex surface of a ring is an example of a toric surface. In an eye whose cornea is of this form the image of a point will not be another point, and from this fact the defect is called **astigmatism.** This asymmetry of refraction was first noticed by Dr. Thomas Young, the celebrated physicist, but Sir George Airy, who had the defect in his own eyes, was, in 1827, the first to correct it by means of suitable lenses.

Astigmatism may be produced by faulty curvature or oblique position of the crystalline lens, but its most common source is asymmetrical curvature of

the cornea. If the curvature of the cornea be normal in one meridian and too little or too great in the meridian at right angles to this one, the defect may evidently be corrected by a lens having no curvature in the normal meridian, and having in the meridian at right angles to this such curvature as will counteract the defective curvature of the cornea in this meridian. A lens of this nature would be cylindrical in form. If, with the equalization of curvature, the eye is still ametropic, a spherical lens may be combined with the cylindrical lens. The spherical surface may be ground on one side of the glass and the cylindrical on the other; or a toric curvature may be ground on one side of the glass, leaving the other side plane; and, if desired, this may then be made concave and the lens periscopic. Toric lenses are not much used, as they are more difficult to grind than spherical and cylindrical surfaces.

In a cylindrical lens the line drawn through the summit of curvature and parallel to the axis of the cylinder is called the **axis of the lens.*** Reference to Fig. 34 renders it clear that in the direction at right angles to the axis AA' of the lens a cylindrical lens has the same action that a spherical lens of like radius of curvature and index would have in this

* This must not be confounded with the axis of the *optical system.*

direction, and that the deviating power of the lens is confined entirely to this meridian. Rays of light parallel to the axis OO' passing through such a lens would not be united in a focus. They would, however, all meet a line which is parallel to the axis of the lens, and whose distance from the lens is equal to the focal length of a spherical lens of the same

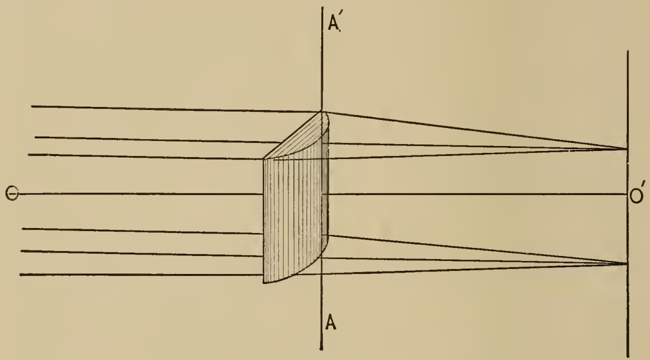

FIG. 34.

radius of curvature and index. This distance is called the **focal distance** of the cylindrical lens, and the line is called the **focal line.** Similarly, rays diverging from a point will, after refraction by the lens, all pass through a line conjugate to the point from which the rays proceed.

If we take another cylindrical lens whose axis is at right angles to that of the first lens, it will bring all

the rays from a point into a line at right angles to the first line.*

In Fig. 35 the rays from the point P all pass through the line AB; and rays from P' all pass through the line $A'B'$ at right angles to AB. Hence, if these two lenses have equal refractive

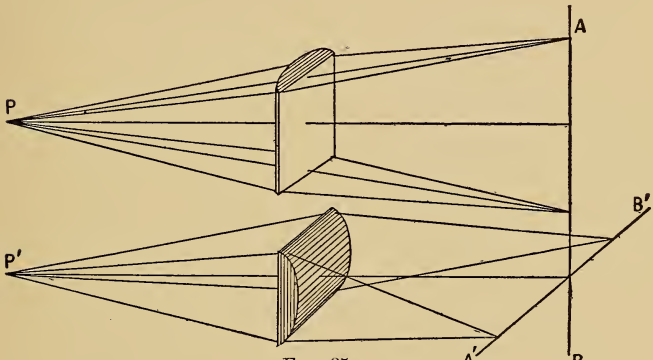

FIG. 35.

power, and if we combine them so that the rays after deviation by the first lens pass immediately through the second lens, the effect of both lenses will be to cause the rays to meet in the *point* where the lines intersect; for the rays must all pass through both lines. In other words, two equal cylindrical lenses placed with their axes at right angles are equivalent

* It will be understood that we are bound by the same restrictions as in the case of spherical lenses, that is, we must suppose that only rays near the axis of the pencil pass through the lens.

to a spherical lens of the same index and radius of curvature. In Fig. 36 let P represent the point at which a ray parallel to the axis of the refracting system meets a spherical lens; then when this ray reaches the focus of the lens it will have been so deviated as to pass through the axis at F. The motion of a point from P to F may be considered as the

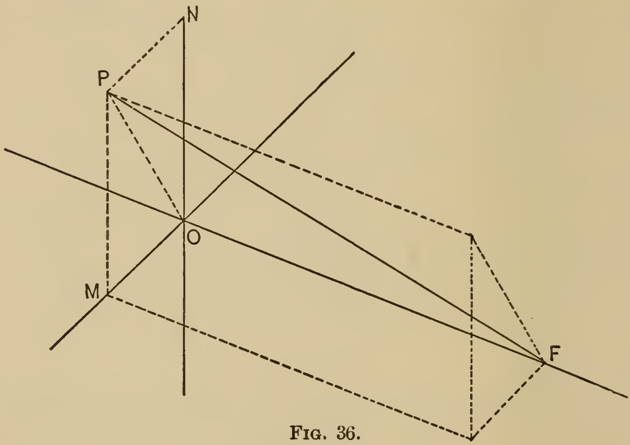

FIG. 36.

resultant of three motions at right angles to each other: 1st, from P to M; 2d, from M to O; and 3d, from O to F. In the case of the spherical lens the deviation in the plane $PMON$ will be represented by the line PO; if instead of a spherical lens we take two equal cylindrical lenses with axes at right angles, then in this plane the ray will be deviated by one lens from P to M, and by the other from M to O.

The result is the same in the two cases. If the two cylindrical lenses have different focal lengths, their combined effect will not be equivalent to that of a spherical lens, because their focal lines will not intersect. If the focal length of the first lens is less than that of the second, it is clear that when a ray parallel to the axis of the refracting system has been brought by the first lens to its intersection with the first focal line, it will not yet have reached its intersection with the second focal line.

Let C (Fig. 37) represent a combination of two cylindrical lenses whose axes are vertical and horizontal respectively. The line AB is conjugate to P as regards the lens whose

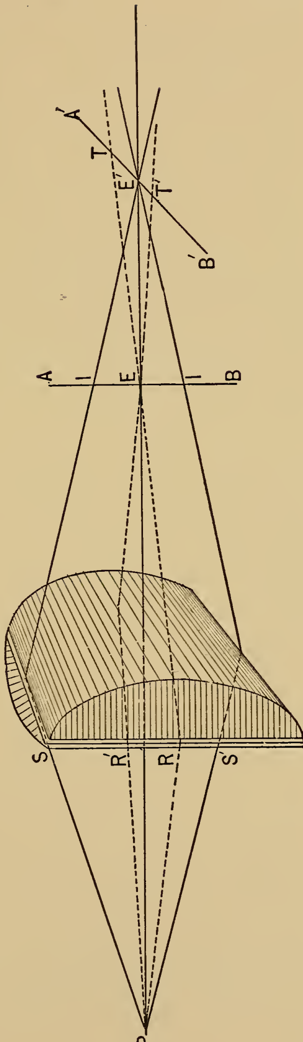

FIG. 37.

axis is vertical; and $A'B'$, at right angles to AB, is conjugate to P as regards the lens whose axis is horizontal. We take two rays, PR and PR', so situated that they pass through the axis of the second lens. Hence they will be deviated only by the first lens; they will intersect at E in the first conjugate focal line, and after crossing at this point they will meet the second conjugate focal line at T and T'. Likewise, if we take two rays, PS and PS', which pass through the axis of the first lens, they will be deviated only by the second lens and will intersect at E' in the second conjugate focal line, meeting the first conjugate focal line before intersection at I and I'. All rays which do not pass through the axis of either lens will be deviated by both lenses, and the deviation by one lens will be superposed upon that of the other. The action of the first lens is such as to cause all rays from P to pass through the vertical line AB conjugate to P; and since the second lens deviates light in the vertical meridian, its effect is to change the position of rays in the line AB, but not to deviate them out of this line; hence all rays from P must pass through AB. Similarly, the action of the second lens is such as to cause all rays from P to pass through the horizontal line $A'B'$, and since the first lens deviates light only in the horizontal meridian, its effect is to change the position of rays in the line $A'B'$, but not to deviate them out of this

line. From this we see that rays proceeding from a point will never be united in a focus by such a lens; they will, however, all pass through two lines which are at right angles to each other, called **focal lines.** If the point is so far distant that the rays may be regarded as parallel, the lines AB and $A'B'$ are the *principal* focal lines. The interval between the principal focal lines is called the **focal interval of Sturm.***

It is clear that the more nearly equal the two cylindrical lenses are in power, the shorter will be the focal interval, and the more closely will the image of a point resemble a point.

Since two equal cylindrical lenses with axes at right angles are equivalent to a spherical lens, it follows that two unequal cylindrical lenses, similarly combined, are equivalent to a sphero-cylindrical lens; for we may regard the unequal lenses as composed of two equal cylindrical lenses with the addition of another cylindrical lens.

In order to investigate the image of a point after refraction by an astigmatic surface, we must suppose a screen to be placed at different positions in the path of the rays so as to intercept them. If a screen be placed at AB, the image of the point P, as formed on the screen, will be the vertical line II'.

* The theory of refraction by asymmetrical surfaces was first demonstrated by Sturm in 1845. See "Comptes Rendus de l'Acad. des Sci. de Paris," tom. xx., pp. 554, 761, 1238.

Likewise, the image of the point as formed at $A'B'$ is TT'. To the left of AB the image will be elliptical, for the rays will not yet have intersected either focal line, but they will be nearer the vertical than the horizontal intersection. To the right of $A'B'$ the image will be elliptical, for the rays have passed both vertical and horizontal intersections, but are nearer to the horizontal than to the vertical intersection. Between AB and $A'B'$ the image will also be elliptical, except in one position, in which the distance of the rays from the axis is the same in the vertical as in the horizontal meridian; it will then be a circle. This is called the **circle of least confusion.** To the left of this circle the ellipse has, in our case, the long axis vertical, and to the right the long axis is horizontal.

The image of a *line* will vary according to its position in relation to the focal lines. If the line be a vertical one passing through P, then at AB every point of the line will have as its image a vertical line such as II'. The image of the vertical line will be a *lengthened* and *intensified* line. At $A'B'$, however, every point of the vertical line will have as its image a horizontal line such as TT', and the image of the vertical line will be an aggregation of horizontal lines; it will therefore be a *broad* and *indistinct* line. At any other point the image will consist of a superposition of confusion ellipses or circles.

If the line be horizontal, we shall have a broad blurred image at AB and a clear intensified image at $A'B'$. If the line be neither vertical nor horizontal, its image will be blurred in all positions.

From what has been said we see that a vertical line appears distinct at the focus of the cylindrical lens whose axis is vertical, and a horizontal line appears distinct at the focus of the lens whose axis is horizontal; or, since the meridians of refraction are at right angles to the axes, a vertical line appears distinct at the focus of the horizontal meridian, and a horizontal line appears distinct at the focus of the vertical meridian.*

The formulæ which we have deduced for spherical lenses are also applicable to cylindrical lenses. We have only to bear in mind that the action of the latter is confined entirely to the meridian at right angles to the axis of the lens. What has been said of the effect of spherical lenses upon the size of retinal images applies, therefore, to cylindrical lenses, with the understanding that this effect is confined to the refracting meridian, no effect being produced by the cylindrical lens in the meridian of its axis.

Astigmatism of the eye is a curvature defect,

* The student who finds it difficult to comprehend refraction through the double cylindrical or sphero-cylindrical lens should construct for himself, or procure from an instrument maker, thread models, which illustrate very clearly astigmatic refraction.

while hyperopia and myopia are, in a large majority of cases, axial defects. The effect of hyperopic or myopic astigmatism upon retinal images will, consequently, not be analogous to that of axial hyperopia or myopia. If the eye is hyperopic in one meridian and emmetropic in the meridian at right angles to this, the defect in curvature in the hyperopic meridian is the same as if a concave cylindrical lens were placed in contact with a normal cornea. The effect of such a lens, since it would be within the anterior focus of the eye, would be to enlarge images in the refracting meridian of the lens. In other words, since the curvature of the eye is less in the hyperopic than in the emmetropic meridian, the anterior focal distance is greater in the faulty than in the normal meridian; and consequently the image of an object will be too large in the former meridian, for the size of the image is proportional to the anterior focal distance. Similarly, in myopic astigmatism, the image is too small in the myopic meridian. But we must remember that the retina is not in the proper position to receive an accurately focused image in the faulty meridian; and, to demonstrate the effect of this faulty position of the retina, let O, Fig. 38, be the optical centre of a lens which may, for the present purpose, represent the eye; then, if A be conjugate to P, the image of PQ will be AB. If, now, the screen or retina remain at

A, while the lens is increased in power so that P' is conjugate to P, the true image, $P'Q'$, will be smaller than AB, but the indistinct image, as intercepted by the screen, will be AC, which is larger than AB.*

This is why a round object such as the full moon appears greater in the meridian of myopic refraction than in the emmetropic meridian, though the accurate image is less in the myopic meridian. In like man-

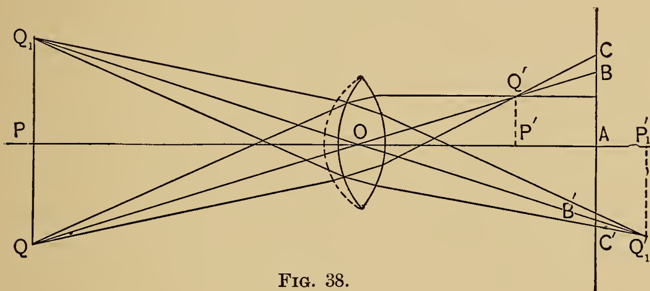

FIG. 38.

ner, if the screen remain at A while the conjugate of P is changed from A to P_1', it is apparent that by the change both the focused and blurred images will be rendered larger than the normal, but the blurred image will be less enlarged than the other.

We can now understand the influence upon retinal images of cylindrical lenses used as spectacles. A proper convex cylindrical lens worn as a spectacle

* The change in position of the optical centre, being too slight to affect the result, is neglected in the figure.

brings the image of an object to an accurate focus on the retina; but it also enlarges the image in the meridian at right angles to the axis of the lens. For, as we have seen in Fig. 38, if the lens be worn at the anterior focus of the eye, the new image will be of the same size as $P_1'Q_1'$, since the effect of the lens is to bring the image forward without changing its size. If the lens be worn without the anterior focus of the eye, then the new image will be larger than $P_1'Q_1'$; in either case it will be larger than the blurred image AC', which the eye receives without the lens, and larger than AB', the normal image. A proper concave cylindrical lens throws the image back upon the retina; it also minifies the image in the refracting meridian of the lens. If the lens be worn at the anterior focus of the eye, $P'Q'$ will represent the size of the new image; and if the lens be worn without this focus, the image will be smaller than $P'Q'$; in either case it will be smaller than the blurred image AC, which the eye receives without the lens, and smaller than AB, the normal image. Hence we see that cylindrical lenses worn as spectacles do not, under any circumstances, produce normal retinal images; all objects are magnified in the refracting meridian by a convex lens, and minified in this meridian by a concave lens. If the lens could be worn in contact with the cornea, the seat of defective curvature, normal images would result. The same position of the

spherical lens would be necessary to produce images of normal size in curvature hyperopia and myopia; but, as shown in the preceding chapter, in ordinary axial hyperopia and myopia, the lens must be placed at the anterior focus of the eye in order that images of normal size be produced.

CHAPTER IX

THE TWISTING PROPERTY OF CYLINDRICAL LENSES

As a consequence of the effect of cylindrical lenses upon retinal images, it follows that if one hold such a lens in front of the eye, and through the lens look at a distant rectangular object as a picture frame or test-type card, there will be observed a distortion of the object, which will vary with every variation in the position of the lens. If the axis of the lens be parallel to one of the sides of the object, the rectangular form of the object will be retained, but the ratio of the sides will be altered. The side which is parallel to the axis will not be changed, while that which is perpendicular to the axis will be increased or diminished. If now the lens be rotated in its own plane, the distortion will no longer be confined to the apparent size of the object ; it will also affect the direction of the lines forming the sides, so that the rectangular object will assume the form of an oblique parallelogram. Use is made of this phenomenon to determine the position of the axis of a cylindrical lens. Looking through the lens at a distant straight line, we observe the position of the lens in which

there is no apparent deviation of the line; the axis of the lens must be either parallel or perpendicular to this line.

We have seen that a cylindrical lens has in its refracting meridian the same effect that a similar spherical lens would have. The formula by which the magnifying or minifying power is obtained has been given in Chapter VII. (page 112).

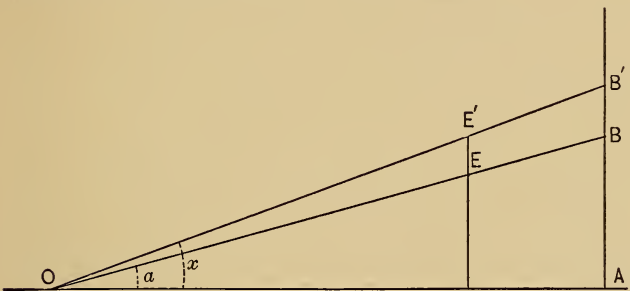

Fig. 39.

Let OA (Fig. 39) be a line parallel to the axis of a cylindrical lens, and let BA be perpendicular to the axis. If we look through the lens at the line OA, the image of this line on the retina will be the same as without the lens. If we look at the line AB, and if the lens be convex, then AB will be magnified in its own direction, and will appear as the line AB'. If we now look at an oblique line OB, its direction will be changed, for B will appear at B' and E will appear at E', and so on for every

other point of the line. The line OB will consequently assume the position OB'. Hence any line not parallel or perpendicular to the axis of the lens undergoes an angular deviation when viewed through a cylindrical lens. If a is the angle which the line makes with the axis, and x the angle which the line appears to make with the axis, we have

$$\tan a = \frac{AB}{OA}; \quad \tan x = \frac{AB'}{OA};$$

from which
$$\tan x = \frac{AB'}{AB} \tan a.$$

$\frac{AB'}{AB}$ being the magnifying power of the lens may be called m. Then $\tan x = m \tan a$. When the lens is concave m is less than unity, and the line OB' appears as OB.

Taking a convex lens, we look through it at a pencil held at arm's length from the eye. When the pencil is parallel to the axis of the lens there is no apparent deviation, but as we turn the pencil through an angle a it appears to make the greater angle x with the axis of the lens. At first x increases more rapidly than a, and $x - a$, which is the angle made by the apparent position with the real position, becomes rapidly greater. As the pencil is turned farther, the angle $x - a$ changes more slowly, then comes to a standstill; and finally a increases

more rapidly than x, and $x-a$ diminishes, so that when the pencil has been turned through 90 degrees a has overtaken x and there is no deviation. If for the pencil we substitute a distant straight line and turn the lens while the line remains stationary, we shall have the same effect. As we turn the lens to the right, the line appears deviated to the left, since the apparent position always makes a greater angle with the axis than the real line. After reaching a maximum deviation the apparent line now travels backward, coinciding with the real position when the lens has been turned through 90 degrees. If we take a concave lens, m being less than unity, the angle which the apparent position of the line makes with the axis will be less than that which the real position makes; consequently, when we look at a distant line and turn the lens to the right, the apparent position also moves to the right, and after reaching a maximum deviation it travels backward, coinciding with the real position when the lens has been turned through 90 degrees.

This to-and-fro deviation, familiar to every one who uses the oculists' trial lenses, vanishes under certain conditions when we experiment with the convex lens. If we take a convex cylindrical lens of four dioptres and, holding it about $\frac{1}{5}$ metre in front of the eye, look at a vertical line across the room, we shall have, upon turning the lens, the to-and-fro

motion of the line; but as we move the lens farther from the eye, the line becomes so indistinct that we cannot determine its movement; continuing to increase the distance between the eye and lens, we now notice that as we turn the lens the behavior of the observed line has entirely changed. It no longer moves to and fro, but as the axis of the lens is turned through 90 degrees the apparent position of the line moves through twice this angle, or 180 degrees. This phenomenon can be most easily observed by holding a pencil or similar object at arm's length and viewing it through a cylindrical lens of ten or twelve dioptres held about $\frac{1}{3}$ metre in front of the eye.

To explain this we shall compare the action of the cylindrical lens with that of the spherical lens of the same radius of curvature and refractive index. It has been shown that if a convex spherical lens be held before the eye at a sufficient distance, an inverted image of an object will be formed in the air in front of the eye, and a second image will be formed by the eye.

In Fig. 40 let (1) represent a rectangular object such as a picture frame; then if it be viewed through a spherical lens held beyond its focal length from the eye, (2) will represent the object as it will appear to the observer. If we replace the spherical lens by a similar cylindrical lens with axis vertical, (3)

will represent the object as it will appear to the observer. The cylindrical lens has the same effect as the spherical one in deviating the rays in the meridian at right angles to the axis of the lens, that is, rays from the right of the object are made to cross over and intersect on the left, and *vice versa*. The object therefore appears reversed in this direction; but it is not reversed in the meridian parallel to the axis of the lens; in this meridian the rays are unaffected by the lens. Similarly, (4) represents the

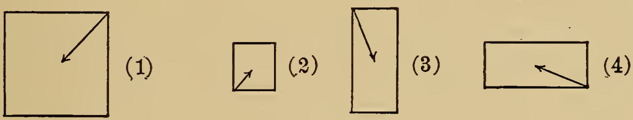

FIG. 40.

object as it would appear when viewed through a cylindrical lens with axis horizontal.

The image as formed with the cylindrical lens will be much blurred, because the rays will, in one meridian, enter the eye diverging from the aerial image; in the meridian at right angles to this they have been unaffected by the lens, and consequently diverge from the more distant object. They cannot, therefore, be accurately focused on the retina.

To this reversal of the object in one meridian is due the apparent rotation of a line through

180 degrees, while the lens is turned through 90 degrees.*

In Fig. 41 let OB represent a line making the angle a with the line OA, which is parallel to the axis of the lens. From what we have just shown, it

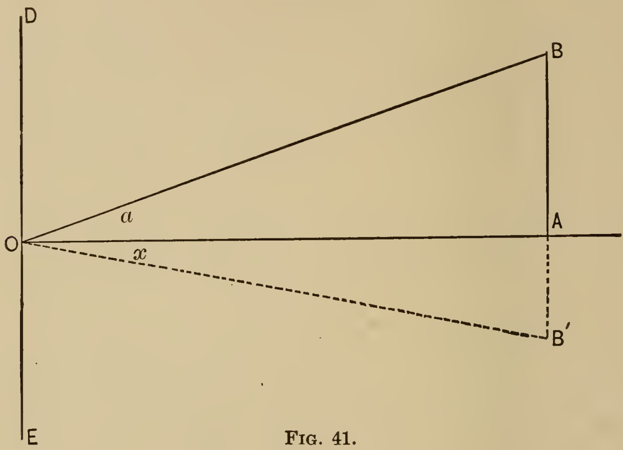

FIG. 41.

is clear that if we view the figure AOB through a convex cylindrical lens so placed as to form the aerial image, then AB will appear in the position AB', and OB will appear at OB'. Hence, if we suppose the original position of the axis to be parallel to OB, in this position there will be no apparent deviation of

* Dr. Carl Koller, the discoverer of the anæsthetic property of cocaine, has demonstrated this phenomenon by means of analytical geometry. See Graefe's "Archives," 1886.

the line OB, but when the axis of the lens is turned into the direction OA, the apparent position of OB is changed from OB to OB'. The angle AOB', which the apparent position of the line makes with the axis of the lens, is as before x. From the figure it is seen that $\tan x = \frac{AB'}{AB} \tan a$; where $\frac{AB'}{AB}$ represents the magnifying power of the lens at right angles to its axis. Hence, $\tan x = m \tan a$. This is the same equation as that derived for the to-and-fro deviation, but the angle which the apparent position OB' makes with the real position OB is now equal to the sum of x and a, whereas in the former case it was equal to their difference. Since m becomes negative when the reversed aerial image is formed, x is also negative, and adding a negative value is equivalent to subtracting a positive value. When a is zero, x is also zero; as a increases, the negative value of x increases; and when a is equal to 90 degrees, x is equal to -90 degrees. That is, if we turn the axis through 90 degrees from OD to OA, in so doing the line OD will be apparently rotated through 180 degrees and will appear in the position OE.

It was shown in Chapter VII. that the reversed image of a distant object is formed when the distance of the lens from the anterior focus of the eye exceeds the focal distance of the lens, but that as the object approaches the lens the distance between the eye and

lens must be increased in order that the negative image be formed. Hence, if a pencil or similar object be held so that we have the continuous deviation through 180 degrees, and if the pencil be then moved nearer the lens, a point will be reached at which this phenomenon will disappear, and upon further approximation of the pencil to the lens, the to-and-fro deviation will be seen. Since spectacle lenses are never worn beyond their focal distance from the eye, it is clear that, so far as this continuous deviation is concerned, nothing analogous occurs in the use of cylindrical spectacles.

As was shown in the preceding chapter, retinal images in astigmatic eyes are not normal in their proportions; we have seen that the effect of astigmatism upon images is similar to that of a cylindrical lens placed in contact with a normal cornea. A cylindrical lens so placed would have a magnifying or minifying effect on images in the refracting meridian of the lens; and, since upon this property depends the apparent deviation of lines, it is clear that in astigmatic eyes all lines not parallel or perpendicular to the axis of the astigmatism are twisted out of their proper relations. A rectangle whose sides do not correspond in direction with the meridians of greatest and least refraction, appears as an oblique parallelogram. This distortion is, however, slight, for the dioptric power of the eye is great in

comparison with the amount of astigmatism. The defect is not appreciable to the person whose eyes are astigmatic, even if the astigmatism is of high degree; but when the astigmatism is corrected by a suitable lens, complaint is frequently made of annoying distortion of lines. This annoyance is fortunately transitory. Since a cylindrical lens, as worn before the eye, cannot reduce the retinal image to its proper proportions, it is evident that it cannot correct the distortion of lines; furthermore, by recalling what was said on this subject in the preceding chapter, it will be seen that the effect of the lens is to increase the distortion in hyperopic astigmatism; in myopic astigmatism the distortion produced by the lens is in the opposite direction to that existing in the *blurred* image without the lens. In either case it is easy to see why annoyance should arise when glasses are first worn.*

* Aside from this actual distortion of the retinal image, there frequently occurs a distortion due to mental influence. For the explanation of this peculiar phenomenon the following articles may be consulted:

"The Effect of a Cyl. Lens with Vert. Axis Placed before One Eye," by O. F. Wadsworth, *Trans. American Ophth. Soc.*, 1875; "Binoc. Metamorphopsia," *Archives of Ophthalmology*, 1889, and "New Tests for Binoc. Vision," *Trans. American Ophth. Soc.*, 1890, by J. A. Lippincott; "Stereo. Illusions Evoked by Prismatic and Cyl. Spec. Glasses," by John Green, *Trans. American Ophth. Soc.*, 1889.

CHAPTER X

THE SPHERO-CYLINDRICAL EQUIVALENCE OF BI-CYLINDRICAL LENSES

It was shown in Chapter VIII. that two cylindrical lenses whose axes are at right angles are equivalent to a spherical lens if the two lenses have the same dioptric power, and to a sphero-cylindrical lens if the two lenses differ in power. It remains now to find the equivalent of any two cylindrical lenses whose axes are not at right angles. Formerly it was not uncommon for oculists to prescribe glasses consisting of a combination of two cylindrical lenses obliquely inclined to each other. This was done notwithstanding that attention had already been called to the fact that such a combination is equivalent to a sphero-cylindrical lens. Sir G. G. Stokes, a celebrated English physicist, first demonstrated this problem, at least for the special case in which two lenses of equal but opposite curvature are combined. He devised this combination as a test for astigmatism. The two lenses are placed the one over the other so that by rotating one lens any angle between the axes can be obtained. This

combination is called the **Stokes lens.** To use it as a test for astigmatism it should be placed before the eye to be examined, and the angle between the axes varied until the position of best vision is found, when further improvement can usually be obtained by adding a spherical lens. From a table constructed by calculation the amount of astigmatism can be deduced.*

Donders in his book on refraction refers to Stokes' demonstration, and also presents a solution applicable for any two cylindrical lenses; but while his conclusion is correct, the demonstration is defective; for, as will be subsequently shown, his assumption is not generally true. In 1886 Dr. Jackson, of Philadelphia, read before the American Ophthalmological Society a complete demonstration of the problem of cylindrical refraction, and at the same meeting Dr. Gustavus Hay, of Boston, offered a somewhat different solution.† In 1888 Prentice‡ published a solution, similar in principle to those of Jackson and Hay, and in 1893 Dr. Weiland,§ of Philadelphia, published a solution based upon that of Donders, but

* The "Stokes lens" is chiefly of historical interest; it is rarely if ever used as a test for astigmatism at the present day, for other more convenient tests have supplanted it.

† *Trans. American Ophth. Soc.*, 1886.

‡ *Dioptric Formulæ, Cylindrical Lenses*, 1888.

§ *Archives of Ophthalmology*, Vol. XXII., No. 4, and Vol. XXIII., No. 1.

this, containing the same error as that of Donders, is not a general solution. There is also given in Heath's "Geometrical Optics" * a solution by means of analytical geometry, which leaves nothing to be desired except that this method is unsuitable for the use of students who are not familiar with the higher mathematics. Thus we see that this question has not lacked investigation. With our more accurate methods of examination it is of less practical importance than formerly, for it is seldom necessary in testing the vision of an eye to resort to two cylindrical lenses obliquely inclined; moreover, if, in any case, it should be found convenient to place two lenses in this manner, the sphero-cylindrical equivalent can be found without formulæ or calculation. Nevertheless this subject must always be of interest to the oculist who wishes to have a scientific knowledge of optics, and therefore, before giving the practical method of determining the equivalent in any case, we shall demonstrate that any two cylindrical lenses acting in combination are equivalent to two other cylindrical lenses whose axes are at right angles to each other, and consequently to a sphero-cylindrical lens. Our apology for adding still another solution to the list of those already published is that in all these solutions the process of

* "Geometrical Optics," Heath, 2d ed., p. 186.

eliminating the special point at which the ray meets the lens is unnecessarily tedious.

In studying refraction by two cylindrical lenses at right angles to each other, we learned that a ray of light in its progress from a point, P, on a spherical lens, Fig. 36, to its intersection with the axis at the focus, F, undergoes the same change in position as if it were first moved from P to O in the plane of the lens, and then moved along the axis from O to F. Furthermore we learned that the change of position from P to O, in the plane of the lens, is identical with the resultant of the two motions from P to M and from M to O, which would be produced by two equal cylindrical lenses at right angles to each other. In the same way we may study the result of the deviation of a ray of light by two cylindrical lenses obliquely inclined. In Fig. 42 let AO and BO represent the axes of the two lenses; then a ray perpendicular to the plane of the lens, and meeting the lens at P, would, if acted upon by the first lens only, be so deviated in the meridian, PM, as to intersect the principal focal line of the lens at the distance F from the lens. The motion of the ray from A to F (F is not represented in the figure) is equivalent to the resultant of the two motions from P to M and from M to F. Similarly this ray, if acted upon by the second lens only, would be deviated in the meridian PN, and would intersect the focal line of

this lens at a distance F_1 from the lens. Now each of these displacements, PM and PN, in the plane of the combined lenses, is equivalent to two displacements at right angles;* thus the displacement from P to M is the resultant of the displacements from

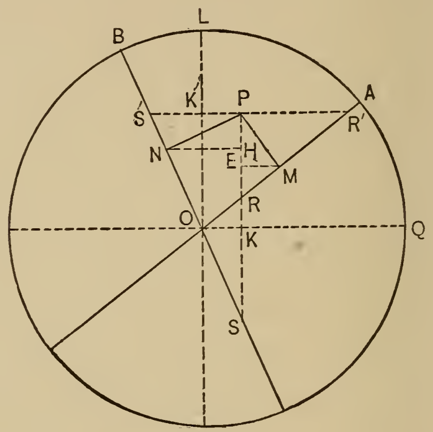

FIG. 42.

P to E and from E to M, and the displacement from P to N is the resultant of the displacements from P to H and from H to N. If the two lenses act in combination, the deviation produced by one lens must be superposed upon that produced by the other. Each lens produces a certain deviation in

* Since we neglect the thickness of the lenses, they both lie in the same plane.

the direction PK or LO, and also a deviation in the direction PK' or QO, at right angles to LO. If we add the deviation which the first lens produces in the direction LO to that which the second lens produces in this direction, the result is the deviation in this direction which the two lenses would produce acting in combination; and this same deviation might evidently be produced by a certain prism whose edge is perpendicular to LO, or by a certain cylindrical lens whose axis is represented by QO. In the same way the deviation produced by the two lenses in the direction QO might also be produced by a cylindrical lens whose axis is represented by LO.

From this we see that any two cylindrical lenses have upon a ray of light the same deviating effect as two other lenses whose axes are at right angles. This is true for any point on the lens and for any position of LO and QO; but we do not mean that the *same* two lenses at right angles are equivalent to the obliquely inclined lenses for different positions of the point P. The problem that we wish to prove is that for a *particular position* of the lines LO and QO the *same* two cylindrical lenses are equivalent to the obliquely inclined lenses irrespective of the position of the point P at which the ray meets the lens.

If the focal length of the two lenses were equal, we should find the combined action of the two lenses

in the direction PK by adding the displacement PE, produced by the first lens, to PH, produced by the second lens; but when the strength of the lenses is not the same, these distances do not measure the displacements which the two lenses produce in the same time.*

Hence we must find an expression from which we can reckon the deviating power of the two lenses acting through the same distance. Since all rays parallel to the axis of the refracting system are refracted to the focus or focal line of the lens, it is evident that the displacement which any ray undergoes in the plane of the lens is proportional to the distance from the axis at which the ray meets the lens; and so long as the focal length remains unchanged, this displacement measures the deviating power of a lens at any point on its surface. It is also evident that the time required to produce a certain displacement varies inversely as the focal length of the lens, or directly as the dioptric power. Hence the deviating power of a lens at any point on its surface is measured by the expression $\frac{D}{F}$, in which D represents the distance from the axis at which the ray meets the lens and F the focal length.

Thus the deviating power of the first lens in the

* Compare Chap. I., p. 22.

meridian PM is expressed by $\frac{PM}{F}$, P being any point on the lens; and the deviating power of the second lens for the same point is expressed by $\frac{PN}{F_1}$; or if we replace the focal lengths by the dioptric values, and represent these by C and C_1, respectively, we have $PM \cdot C$ as the measure of the deviating power of the first lens, and $PN \cdot C_1$ as the measure of this power for the second lens.

Let the angle AOB, which is included between the axes of the two cylindrical lenses, be represented by the letter a; and let us assume that in a certain position of LO and QO the two obliquely inclined lenses may be replaced by two other lenses at right angles. The unknown angle LOA, which the axis LO must make with OA, is denoted by x, and the angle LOB, which is equal to $a - x$, is denoted by y. It is readily seen from the figure that PME, MPR', and ORK are each equal to x, and that PNH, $S'PN$ and OSK are each equal to y.

The dioptric value of the assumed lens whose axis is OQ is denoted by C_2, and that of the lens whose axis is LO is denoted by C_3. If the lens C_2 is equivalent to the combined action of the two lenses C and C_1 in the direction LO, we have the equation

$$PK \cdot C_2 = PE \cdot C + PH \cdot C_1, \qquad (1)$$

for $PK \cdot C_2$ expresses the deviating power of the lens C_2 in its refracting meridian PK, and $PE \cdot C$ and $PH \cdot C_1$ express the deviating power of the first and second lens respectively in the direction PK; and, since when both lenses are convex the displacements PE and PH are both toward the axis OQ, we must add the deviating powers of the two lenses to find the equivalent lens.* In the direction PK' or QO the displacements EM and HN are in opposite directions, and the *difference* in power of the two lenses in this direction will express the power of the equivalent lens. Hence we have also the equation

$$PK' \cdot C_3 = HN \cdot C_1 - EM \cdot C. \qquad (2)$$

But $PE = PM \cdot \sin x, \qquad PH = PN \cdot \sin y,$

$$HN = PN \cdot \cos y, \text{ and } EM = PM \cdot \cos x.$$

Hence equation (1) becomes

$$C_2 = \frac{PM}{PK} \cdot \sin x \cdot C + \frac{PN}{PK} \cdot \sin y \cdot C_1.$$

But $PM = PR \cdot \sin x$, and $PN = PS \cdot \sin y$;

also, $PR = PK - RK$, and $PS = PK + KS$.

* We take as the typical case two convex lenses; the same demonstration is applicable if one or both lenses be concave, it being only necessary to change the sign of C or of C_1, or of both.

Hence we have

$$C_2 = \frac{PK - RK}{PK} \sin^2 x \cdot C + \frac{PK + KS}{PK} \sin^2 y \cdot C_1 ;$$

or, $$C_2 = \sin^2 x \cdot C + \sin^2 y \cdot C_1$$

$$- \frac{RK}{PK} \cdot \sin^2 x \cdot C + \frac{KS}{PK} \cdot \sin^2 y \cdot C_1. \qquad (3)$$

If this is true for all positions of P, it must be true when P lies on the line LO. It is readily seen that when P is moved over to K', RK and KS both become zero, and our equation reduces to the form

$$C_2 = \sin^2 x \cdot C + \sin^2 y \cdot C_1 ; \qquad (4)$$

and since C_2 is a constant quantity for all positions of P, if it is an equivalent lens for C and C_1 in its refracting meridian, then the algebraic sum of the last two terms of the second member of equation (3) must be equal to zero.* Thus we have as the condition of equivalence for all points on the lens that the expression

$$- \frac{RK}{PK} \sin^2 x \cdot C + \frac{KS}{PK} \sin^2 y \cdot C_1$$

* Donders and Weiland in their demonstrations have assumed that a lens C is in the meridian PK equivalent to another lens whose dioptric power is expressed by $C \cdot \sin^2 x$. This we readily see is true only for points on the axis LO.

should be equal to zero; or,

$$\frac{RK}{KS} = \frac{\sin^2 y \cdot C_1}{\sin^2 x \cdot C};$$

but we see from the figure that $RK = OK \cdot \cot x$, and $KS = OK \cdot \cot y$. Substituting and replacing the cotangent of these angles by $\frac{\cos}{\sin}$, we have

$$C \cdot \sin x \cdot \cos x = C_1 \sin y \cdot \cos y;$$

or, $$C \cdot \sin 2x = C_1 \sin 2y;$$

or, $$C \cdot \sin 2x = C_1 \sin 2(a - x).$$

By reduction we obtain the equation

$$\tan 2x = \frac{C_1 \sin 2a}{C + C_1 \cos 2a},$$

and from this we know the value which must be assigned to x.

In the same way we find from equation (2),

$$C_3 = \frac{PN}{PK'} \cos y \cdot C_1 - \frac{PM}{PK'} \cos x \cdot C;$$

but $PN = PS' \cdot \cos y$, and $PS' = PK' + K'S'$,

also $PM = PR' \cdot \cos x$, and $PR' = K'R' - PK'$.

Hence

$$C_3 = \frac{PK' + K'S'}{PK'} \cos^2 y \cdot C_1 - \frac{K'R' - PK'}{PK'} \cos^2 x \cdot C;$$

or,
$$C_3 = \cos^2 y \cdot C_1 + \cos^2 x \cdot C$$
$$+ \frac{K'S'}{PK'} \cos^2 y \cdot C_1 - \frac{K'R'}{PK'} \cos^2 x \cdot C.$$

When P lies on the axis OQ, $K'S'$ and $K'R'$ both become zero, and

$$C_3 = \cos^2 y \cdot C_1 + \cos^2 x \cdot C. \tag{5}$$

Hence

$$\frac{K'S'}{PK'} \cos^2 y \cdot C_1 - \frac{K'R'}{PK'} \cos^2 x \cdot C = 0,$$

from which
$$\frac{K'S'}{K'R'} = \frac{\cos^2 x \cdot C}{\cos^2 y \cdot C_1};$$

or, since
$$K'S' = OK' \cdot \tan y,$$

and
$$K'R' = OK' \tan x,$$

$$\frac{\sin y \cos x}{\cos y \sin x} = \frac{\cos^2 x \cdot C}{\cos^2 y \cdot C_1};$$

or,
$$C \sin 2x = C_1 \sin 2y;$$

from which, as above,

$$\tan 2x = \frac{C_1 \sin 2a}{C + C_1 \cos 2a}.$$

Thus we find that the angle LOA is the same in order that C_3 should be equivalent to C and C_1 in the direction OQ as when C_2 is equivalent to these lenses in the direction LO, and therefore it is proved that by giving a suitable value to the angle LOA, we may substitute for the obliquely inclined lenses two other lenses at right angles, or, the equivalent of the latter, a sphero-cylindrical combination. To find the dioptric value of C_2 and C_3 we have only to substitute the values of x and y which we now know in equations (4) and (5). If we add these two equations, we see that

$$C_2 + C_3 = C + C_1,$$

since $\sin^2 x + \cos^2 x$ and $\sin^2 y + \cos^2 y$

are each equal to unity.

In practice it is not necessary to resort to this calculation, as was stated in the first part of the present chapter. The equivalent may be found in the following manner: Place the two lenses in a trial frame with their axes in proper position; and, holding the frame about $\frac{1}{3}$ metre in front of the eye, look through the lenses at a test-type card or other rectangular object distant five metres or more. Rotate the frame in the plane of the lenses until the position is found in which there is no angular deviation of the vertical and horizontal edges of the card;

then move the frame slightly from right to left or *vice versa* until there is no break in the vertical line as seen through the lens and above it. Notice where this unbroken line cuts the trial frame, and thus read off on the frame the angular marking, which gives the position of the axis of one of the equivalent lenses; the axis of the other is at right angles to this. Having found the position of the axes, we next neutralize the meridian of least refraction by means of a spherical lens; adding now a suitable cylindrical lens, with axis in the meridian already neutralized, we neutralize the meridian of greatest refraction. The sphero-cylindrical combination of equal but opposite power to that required for neutralization represents the equivalent of the two obliquely inclined lenses. The accuracy of this method is limited only by the intervals between lenses in the trial case; it is therefore sufficiently accurate for practical purposes. The nearest equivalent which exists in the trial case can be found in a few moments.

In this way also can the lenticular astigmatism of the eye be found.* Suppose, for instance, that the entire astigmatism of the eye, as found by subjective or objective test, is three dioptres, the meridian of

* It was for the purpose of showing how to find the lenticular astigmatism of the eye that Donders gave his solution of the bi-cylindrical problem.

least curvature being 60 degrees from the horizontal line. This can be represented by a convex lens of three dioptres, with its axis at 60 degrees. If the corneal astigmatism be found by the ophthalmometer to be two dioptres, with the meridian of least curvature at 130 degrees, this will be represented by a convex lens of two dioptres with axis at 130 degrees. The lenticular astigmatism is evidently equal to the entire astigmatism less that of the cornea. To find this we place in the trial frame a convex cylindrical lens of three dioptres with axis at 60 degrees, which represents the entire astigmatism. If we neutralize the corneal astigmatism, we have remaining the lenticular astigmatism. The corneal portion is neutralized by a concave lens of two dioptres with axis at 130 degrees. Placing this lens in the trial frame, we have a combination of two cylindrical lenses obliquely inclined, and the astigmatism of this combination represents the lenticular astigmatism of the eye. By neutralizing the combination, we derive the lens equivalent of the lenticular astigmatism and the angle of its axis.

CHAPTER XI

OBLIQUE REFRACTION THROUGH LENSES

In our study of refraction we have considered only pencils of light whose central ray or axis is perpendicular to the refracting surface. Such a pencil is called **direct**. We have also learned that a spherical surface has greater refractive power for rays which meet it at a distance from the axis than for those which pass near the axis. We have been obliged in order to escape spherical aberration to limit our consideration to those rays which do not deviate far from the axis. We are justified in this, since only small pencils can enter the eye.

We shall now investigate the refraction of small pencils, the axis of which is not perpendicular to the refracting surface. Such a pencil is called **oblique**. Refraction by oblique pencils takes place when we look through a tilted lens, as is frequently done in the use of spectacles. The complete analysis of this subject is difficult and requires a knowledge of the higher mathematics, but we can study oblique refraction in an elementary manner, and this will enable us to understand the problems which present

themselves in ophthalmology. A complete solution, so far as relates to tilted spectacle lenses, has been worked out by Dr. John Green of St. Louis;* and a general solution of the problem of oblique spherical refraction is given in Heath's "Geometrical Optics."† From the formulæ derived by these investigations, the exact value of a lens when tilted at any angle can be found. We shall suppose the lens to be tilted only in the meridian of vertical

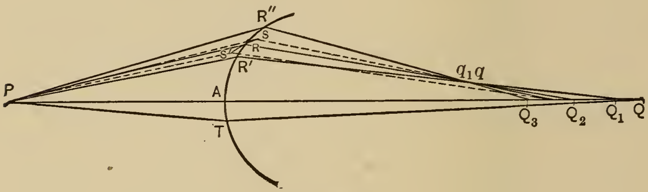

FIG. 43.

refraction. Let A (Fig. 43) represent the principal point of a refracting surface. The direct pencil, whose axis is PA and whose peripheral ray is PT, will after refraction meet the axis at Q. Let $R'PR''$ represent a pencil meeting the surface obliquely, PR being the axis of this pencil. We have learned from our previous studies that it is sufficient to determine the deviation of light in two meridians at right angles to each other. We shall

* *Trans. American Ophth. Soc.*, 1890.
† Heath's "Geom. Optics," 2d ed., p. 179.

therefore investigate the refraction of the oblique pencil first in the vertical meridian or plane of the paper, as represented by $R'R''$, and then in the horizontal meridian, as represented by SS'. We may consider the rays PR', PR, and PR'' as rays of the direct pencil which are so far from the axis that spherical aberration cannot be neglected. These rays will after refraction meet the axis at Q_1, Q_2, and Q_3, respectively. The refracted rays $R'Q_1$ and RQ_2 will meet at q; RQ_2 and $R''Q_3$ will meet at q_1. If the pencil be small, q and q_1 will be so near to each other that we may regard them as identical. Then the point of intersection q will be the focus of the pencil in the vertical meridian, and Rq will be the focal distance in this meridian. If now we take the rays PS and PS' in the horizontal meridian, it is evident that after refraction they will meet the axis at Q_2; for their vertical distance from A is the same as that of the ray PR, the axis of the pencil. Hence Q_2 is the horizontal focus, and RQ_2 is the horizontal focal distance. From this we see that oblique spherical refraction is astigmatic. Since Rq is the focal distance in any vertical meridian, all rays of the pencil must intersect a straight line passing through q and parallel to SS'.* Likewise all rays must have their

* SS', being a small arc, does not materially differ from a straight line.

horizontal intersections on the line Q_1Q_3; and since the pencil is small, the error will be inappreciable if we replace Q_1Q_3 by a line through Q_2 parallel to $R'R''$. These lines drawn through q and Q_2 are the focal lines of the pencil, and qQ_2 is the focal interval. It is clear that the focal interval increases as the distance of the oblique pencil from the axis PA increases. As AQ is the focal distance for the direct pencil, it is seen that by the change from direct to oblique refraction the focal distance is

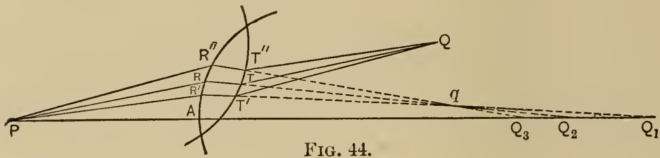

FIG. 44.

shortened in both vertical and horizontal meridians, but more in the former than in the latter; for AQ_2 and Rq are both less than AQ, and Rq is less than AQ_2. In the question before us we have oblique refraction at two surfaces. We shall consider only the case in which the axis of the pencil passes through the optical centre of the lens, so that after the two refractions its direction is parallel to that before refraction; and we shall disregard the lateral displacement due to the thickness of the lens.

In Fig. 44 let $R'PR''$ represent the vertical section of a pencil meeting the lens obliquely at

$R'R''$; then since the axis of the pencil, PR, passes through the optical centre of the lens, its direction after emerging from the lens will be parallel to PR. Let Q be conjugate to P in the vertical meridian. We have seen the refractive power of the first surface is increased by tilting the lens; consequently a shorter incident pencil $R'PR''$ will cause the refracted pencil to assume the convergence $R'qR''$ than if the pencil were direct. Likewise if we suppose a pencil $T'QT''$ to proceed from Q so that after refraction it assumes the divergence $R'qR''$, appearing to proceed from q, the pencil $T'QT''$ will be shorter than if it were direct. Since the path of light is reversible, the same will be true when $T'QT''$ is an emergent pencil; in other words both the incident and emergent pencils are shortened by tilting the lens, and the lens is increased in refractive power. The same reasoning applies also to the refraction in the horizontal meridian, but in this meridian the increase of power is less at each surface than in the vertical meridian. Hence a spherical lens tilted in its vertical refracting meridian is equivalent to a sphero-cylindrical lens. Reference to Fig. 43 will show that as the tilting is increased, the amount of astigmatism increases very rapidly.

If a cylindrical lens whose axis is horizontal be tilted in its vertical refracting meridian, the increase

in power will be the same as that of a similar spherical lens in the vertical meridian. If the axis of the lens be vertical and it be tilted in this meridian, the increase in power will be the same as that which a spherical lens undergoes in the horizontal meridian when tilted in the vertical meridian.

We should learn two practical points from this study: first, the necessity of giving the proper inclination to spectacles, especially when strong lenses are used; and, secondly, the uncertainty of effect of very weak cylinders in combination with strong spherical lenses. A spherical lens of four dioptres acquires more than one-quarter of a dioptre of astigmatism by tilting it through an angle of 15 degrees. Hence a convex cylindrical lens of .25 D., axis vertical, combined with a convex spherical lens of 4 D., would be completely nullified by a slight amount of tilting; on the other hand, if the axis of the lens be horizontal, its action will be practically doubled by the same tilting.

CHAPTER XII

THE EFFECT OF PRISMATIC GLASSES UPON RETINAL IMAGES

In Chapter I. we considered refraction of *rays* through prisms. We must now investigate in an elementary way the more difficult subject of refraction of *pencils* of light such as enter the eye. The study of the effect of prisms upon stereoscopic vision belongs to physiology, and is discussed in treatises on physiological optics. We shall confine our attention to the influence of prisms upon the size and form of retinal images.

For convenience we repeat the following :

1. A principal plane of a prism is a plane perpendicular to the edge of the prism, and therefore to each face of the prism.*

2. By the law of refraction, the incident and refracted rays and the normal to the surface all lie in the same plane.

3. From the equation $\sin i = n \cdot \sin r$, it follows that the greater the angle of incidence, the greater is

* The principal plane of a prism bears no analogy to the principal plane of a spherical refracting surface.

the deviation of the ray ; and the greater the angle of incidence, the greater is the increase in deviation for a fixed increase in the angle of incidence, the rate of change in deviation increasing very rapidly as the angle approaches 90 degrees.

From (1) and (2) we see that a ray which enters the prism in a principal plane must lie in this plane after emergence.

A ray which enters in any other plane passes out of the prism in a plane parallel to that in which it

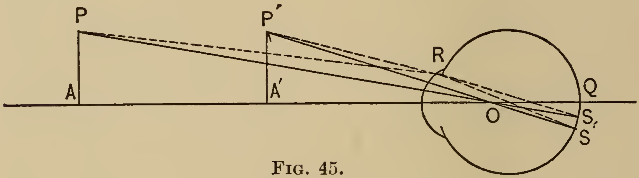

Fig. 45.

entered, the amount of separation between the planes depending upon the thickness of the prism ; and the deviation which this ray undergoes in the principal plane is greater than if the ray were in this plane.*

The effect which these facts have upon images will appear subsequently ; we must first examine the relation between the length of the pencil and the size of the image. Helmholtz' formula applies here as in all cases, but this relation is indicated clearly in Fig. 45. Let *AP* represent the linear dimension of an

* For the demonstration of this see Appendix II.

object, then QS represents its image on the retina. The distance from P to the cornea is the length of the pencil before refraction, RPO is the angle of divergence of the pencil, and AOP or QOS is the *visual* angle which the object subtends.

Thus we see that as the incident pencil becomes shorter the image on the retina increases. If, by any means, rays of light from an object AP are rendered more divergent, so that the pencils appear to proceed from $A'P'$, the image will be enlarged from QS to QS'.*

The effect of prisms upon the length of pencils varies greatly with position of the prism as regards the light which passes into it. If the prism be of small refracting angle, all rays which pass through it near the position of minimum deviation will undergo practically the same amount of deviation.† Small pencils passing through such a prism in this position undergo no change in length, for if the rays all have the same amount of deviation their relative divergence will be unaltered in their passage through the prism. But all other pencils will be altered in length; and to prove this let Fig. 46 represent a principal section of a prism. From O draw ROR', representing a pencil in the position of minimum deviation, and also SOS' and TOT', two pencils, the

* We neglect as inappreciable the slight change in position of the optical centre O, due to change in refractive state of the eye upon approximation of the object. † Chap. I., p. 21.

former being nearer the apex and the latter nearer the base of the prism than ROR'. As has been stated, the pencil ROR' undergoes no change in length. By the first refraction the pencil SOS' is increased in length, for OS makes a greater angle with the normal to the surface than does OS'; its deviation is consequently greater and the divergence of the pencil is diminished. By the second refraction the divergence of the pencil is for a similar

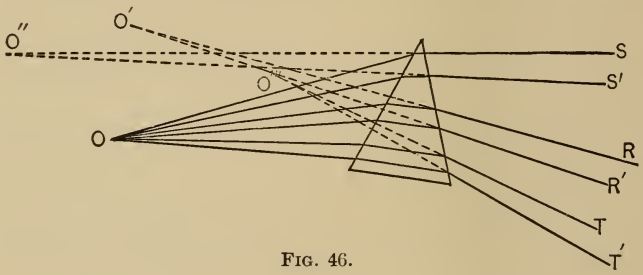

FIG. 46.

reason increased; but since the angles of refraction are greater at the first surface, the difference in the amount of deviation undergone by OS and OS' is greater at this than at the second surface; consequently the lengthening of the first surface outbalances the shortening at the second surface. In the pencil TOT' the angles of refraction are greater at the second surface, and the shortening at this surface outbalances the lengthening at the first surface. Furthermore it will be observed that pencils proceed-

ing from a point will not, after passing through a prism, appear to come from a point; but if the axis of the pencil lie in a principal plane of the prism, as we suppose in Fig. 46, rays proceeding from O will appear to come from two focal lines parallel and perpendicular, respectively, to the edge of the prism. This follows from the analogy to astigmatic refraction at cylindrical surfaces; for the rays of the pencil will not be deviated in the direction of the edge of the prism, the alteration in length taking place in the meridian at right angles to this edge. Since we deal only with small pencils, we may for our present purpose ignore the astigmatic effect and regard the pencil SOS' as proceeding from a point O'' after refraction by the prism. An object seen by pencils such as SOS' appears minified in the direction of the principal plane, since the incident pencils as received by the observer's eye are lengthened; on the other hand, an object seen by pencils such as TOT' appears magnified in this direction, since the incident pencils are shortened.

From this we see that as the apex of a prism is turned toward the object, the image is magnified in the principal plane of the prism; and as the base of the prism is turned toward the object, the image is minified in this plane.*

* Upon this principle is constructed a toy,—the "laughing camera,"—which is sold on the streets and in the shops.

Let us now study the effect of a prism placed before the eye with its principal plane horizontal. If the prism have a refracting angle of 15 or 18 degrees, objects in the field of view will undergo marked distortion. This is due not only to the actual distortion of the retinal image, but also, in part, to mental impressions of previous experience.

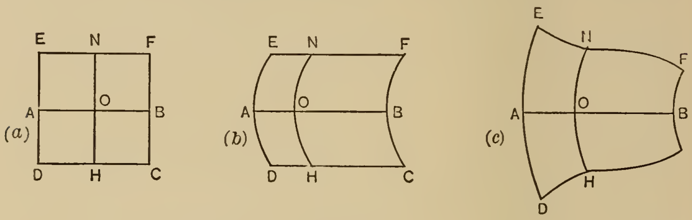

FIG. 47.

When prisms are worn before both eyes there is still further confusion arising from altered convergence and perspective.*

In Fig. 47 (*a*) represents a square object so placed that *O*, its middle point, is seen by rays passing through the prism with minimum deviation; then if the base of the prism is placed to the right, (*b*) is a representation of the object as it appears on the

* Articles bearing upon this subject have been published by a number of writers. Reference may be made to the following: Helmholtz' "Physiological Optics"; "Stereoscopic Illusions Evoked by Prismatic and Cylindrical Spectacle Glasses," John Green, *Trans. American Ophth. Soc.*, 1889.

retina. The portion of the figure near O is unaffected by the prism, but the portion toward the base of the prism is minified, and that toward the apex is magnified in the horizontal meridian. The point A is displaced toward O, and any other point of the line AE is displaced in this direction to a greater extent than is A, because a ray not in a principal plane is more deflected than a corresponding ray lying in this plane. As this increase in deviation continues at an increasing rate with the increase of the distance of the point from A, the line AE appears curved. For the same reason ON and BF appear curved.

Neglecting, as we do, the thickness of the prism, there is no displacement at right angles to AE, and therefore EF is unchanged in direction in the retinal image. Nevertheless EF does seem to be changed in such manner as to assume the form represented in (c) — a change which is due to illusion. The line AE, as seen through the prism, appears at an increased distance while its image is unchanged; consequently the line seems to be increased in length. For a similar reason BF appears shortened. This increasing apparent distance, as the attention is directed farther to the left, also causes this part of the figure to assume a convex cylindrical form; while that portion toward the extreme right appears concave. This effect can be obtained by looking

through a strong prism at the opposite wall of a room, turning the prism so as to get the extreme magnifying and minifying power. Since the lower half of the figure is a reproduction of the upper half, it needs no explanation.

Similar distortions and illusions must be produced by weak prisms, though in a less marked degree; it is not surprising therefore that unpleasant sensations arise when persons attempt to wear comparatively strong prismatic glasses as spectacles. In many cases, however, this disturbance passes away after the glasses have been worn for a short time.

CHAPTER XIII

THE REFLEXION OF LIGHT

The law of reflexion — namely, the angles of incidence and reflexion lie in the same plane and are equal — was, as we have stated, known to the ancients.

As we have found the law of refraction to follow as a necessary consequence of the wave theory of light, being due to the varying velocity of light in different media, so also the law of reflexion corroborates this theory. In reflexion the light does not pass out of the first medium; its direction, however, is reversed. Hence, if we make n equal to minus 1 in any equation pertaining to refraction, we ought to get the corresponding equation for reflexion. By making this substitution we arrive at identical formulæ with those obtained from independent geometrical construction.

The phenomena of reflexion, so far as they concern the ophthalmologist, are exceedingly simple.

The illumination of the interior of the eye is accomplished by means of a mirror, which reflects light from a flame, while the observer is so situated as to

be in the path of the light returning by reflexion and refraction.

If a plane mirror is used, light is thrown into the eye by pencils which appear to come from behind the mirror and from a distance equal to that of the

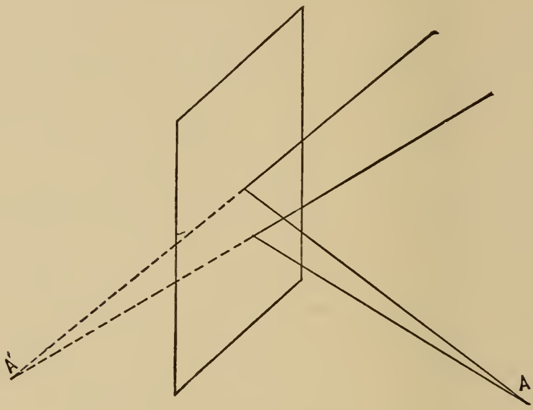

FIG. 48.

lamp from the mirror. This is illustrated in Fig. 48. It can easily be proved that the angles A and A' are equal.

If a concave mirror is used, the light appears to come from a point in front of the mirror. This we deduce from equation (a), page 30. If we make n equal to minus 1 in this equation, the result is an equation expressing the relation between the conju-

gate foci after reflexion at a spherical surface. Making this substitution, equation (a) becomes

$$\frac{1}{f}-\frac{1}{f'}=-\frac{2}{r}, \text{ or } \frac{1}{f'}=\frac{2}{r}+\frac{1}{f};$$

f is the distance of the flame from the mirror, f' is the distance of its conjugate from the mirror, and r is the radius of curvature. Since the mirror is concave, r is negative. Hence, f being positive, f' is negative when $\frac{2}{r}$ is greater than $\frac{1}{f}$ and positive when $\frac{2}{r}$ is less than $\frac{1}{f}$.

In other words, if f is greater than $\frac{r}{2}$, the two conjugate foci lie on the same side of the mirror; if f is less than $\frac{r}{2}$, both f and f' are positive, and consequently lie on opposite sides of the mirror.

The radius of curvature of the concave mirrors used in ophthalmoscopy does not exceed $\frac{1}{2}$ metre; and as the distance of the flame from the mirror is greater than half this radius, the point from which light proceeds is in front of the mirror. The illumination of the eye by the concave mirror is therefore more intense than that by the plane mirror.

Figure 49 illustrates reflexion by the concave mirror; light from A is focused at A'. If A rep-

resent a gas flame, an image of this flame will be formed at A'.

Continuing the study of reflexion at a spherical surface, we see that if we make f equal to infinity, f' is equal to $\frac{r}{2}$, that is, the principal focal distance of a spherical mirror is equal to one-half of the radius of curvature.* If the mirror is concave, $\frac{r}{2}$ is negative, and the principal focus is half way be-

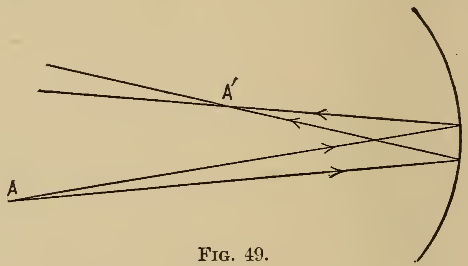

FIG. 49.

tween the centre of curvature and the surface of the mirror. If the mirror is convex, $\frac{r}{2}$ is positive, and the principal focus lies behind the mirror. Clearly such a focus must be virtual, while the negative focus of the concave mirror is real. Hence if we use equation (a), real foci in reflexion are negative while virtual ones are positive.

* It will be observed that in reflexion the two principal foci coincide.

It must be borne in mind that the equation applies only to small pencils near the axis of the surface, for spherical aberration occurs in reflexion as in refraction.

The same relation exists between the size of object and image as in refraction, viz.:

$$\frac{o}{i} = \frac{u}{F}.$$

If u be positive, that is, if the object be without the principal focus, as in Fig. 49, then the image will be positive or negative according as F is positive or negative. When F is negative, as in the concave mirror, the image is real and inverted; when F is positive, as in the convex mirror, the image is virtual and erect.

In the concave mirror u is negative when the object lies nearer the mirror than the principal focus; in the convex mirror u cannot be negative, since the principal focus is virtual and behind the mirror. Hence in reflexion at concave mirrors the image is real and inverted when the object lies without the principal focus; virtual and erect when the object lies within this focus. In reflexion at convex mirrors the image is always virtual and erect.

Since the size of the image is equal to $o \cdot \frac{F}{u}$, the virtual image formed by the concave mirror is larger

than the object, for u, lying between F and the mirror, must be less than F.

The real image formed by the concave mirror is less than the object when u is greater than F, and *vice versa*.

The erect virtual image formed by the convex mirror is always smaller than the object, since u is necessarily greater than F.

Since all refracting surfaces act also as reflecting surfaces, the three refracting surfaces of the eye, namely, the cornea, the anterior, and the posterior surface of the lens, must furnish also three reflecting surfaces. Consequently, when an object is held before the eye, there must be formed by reflexion three images of the object. The first, formed at the convex surface of the cornea, is *virtual* and *erect;* the second, formed at the anterior surface of the lens, is also *virtual* and *erect;* while the third, formed at the concave posterior surface of the lens, is *real* and *inverted.* These images, as seen with the aid of a lighted candle in a darkened room, are of great interest to ophthalmologists. The formation of all three images is conclusive evidence of the presence of the crystalline lens in the eye. Furthermore, by the change in relative size of these images, the increase in curvature of the lens during the act of accommodation can be demonstrated.

Since the size of the image varies with $\frac{u}{F}$, and F is equal to $\frac{r}{2}$, we have a means of measuring the curvature of the refracting surfaces of the eye, provided we can measure the size of the reflected images. A brief description of the way in which this can be done will be given in the next chapter.

CHAPTER XIV

THE OPTICAL PRINCIPLES OF OPHTHALMOMETRY AND OF OPHTHALMOSCOPY

To the genius of Helmholtz we owe the invention of the ophthalmometer, an instrument of great precision for measuring the curvature of the refracting surfaces of the eye. In the construction of this instrument Helmholtz employed a device already in use by astronomers for the measurement of the stars, namely, the production of double images of a single object.

This is possible by means of several contrivances. The simplest is the double prism, such as the Maddox prism found in the oculists' trial case. Helmholtz' device consisted of two plates of glass of equal thickness inclined at an angle, as is shown in Fig. 50. A pencil of light from *o* meets the plate *D*, and is refracted as in the figure. The rays, after emergence, are parallel to their direction before entering the plate, but they undergo a lateral displacement due to the thickness of the glass, so that they all appear to come from *A*. Likewise, that part of the pencil which passes through the plate

E appears to come from B. Hence, if o represent a small object, there will appear after refraction through the plates two similar objects at A and B, respectively.

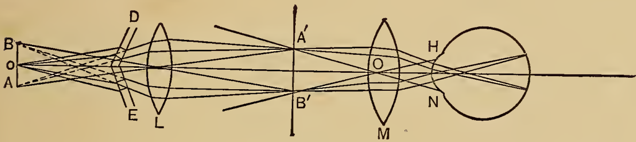

FIG. 50.

If a convex lens be placed at L, a real image of A will be formed at A', and an image of B will be formed at B'. A second convex lens, M, whose principal focal plane is $A'B'$, will render rays from the image $A'B'$ parallel; and the image will be focused on the retina without accommodation.

If the circle whose centre is O (Fig. 51) be viewed through a double prism, or through two inclined plates; and, if the double images are separated to such an extent that the two circles A and B appear to touch at O, it is clear that the amount of separation of the images will be equal to the diameter of the circle O. It is also clear that if the angle between the two plates can be varied, then, by changing this angle to the proper degree, the two

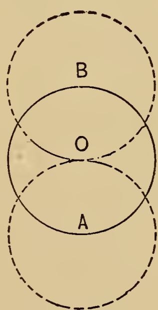

FIG. 51.

images can be made to touch as in the figure. Knowing the angle between the plates, the thickness and refractive index of the glass, the amount of displacement from O to A and from O to B can be calculated.

If the object at O be a reflected image as seen in the cornea, we can obtain its size from the data above mentioned. The size of the object, the size of the image, and the distance of the object from the cornea being known, we deduce the curvature of the cornea from the equation,

$$\frac{o}{i} = \frac{u}{F} = \frac{2u}{r}.$$

The greater the curvature of the surface, the smaller is the image; consequently, if the two images touch in the meridian of least curvature of an astigmatic cornea, they will be separated by an interval in the meridian of greatest curvature; while, if they touch in the meridian of greatest curvature, they will overlap in that of least curvature.

The construction of the modern ophthalmometer of Javal and Schiötz is somewhat different from that of Helmholtz; but the essential optical principles are the same in both instruments.

By the aid of ingenious mechanical devices, observations of the corneal curvature have become a matter of the greatest ease. In the instrument of

Javal and Schiötz the glass plates are replaced by a Wollaston prism, which, like the plates, produces two images of a single object.

Certain crystalline substances possess the peculiar property of **double refraction.** *Iceland spar* is a familiar example of a double refracting substance. Part of the light entering this material undergoes refraction in the ordinary way, while a part possesses the property of having different velocities, and hence different refractive indices in different directions.

This is due to the fact that the constitution of the substance is such as to offer unequal resistance to the passage of light in different directions or axes. The first portion of light which follows the ordinary law of refraction is called the **ordinary** ray; the second portion, whose index varies for different meridians, is called the **extra-ordinary** ray.

If we take a piece of double refracting substance, as Iceland spar or quartz, and through it look at an object placed in such position as regards incident light that the difference in index between the ordinary and extra-ordinary rays causes a separation of these rays, a double image of the object will be formed. Wollaston's prism is based upon this principle.*

* For a detailed account of the phenomenon of double refraction and the construction of Wollaston's prism, consult Preston's "Theory of Light," or other complete treatise on optics.

In the modern ophthalmometer, in which the Wollaston prism is used, the objective L is composed of two lenses separated by an interval; and the prism is placed in this interval between the lenses.

The exact amount of separation which the prism produces in its fixed position is known. The diameter of the object from which light is reflected to the cornea can be varied at will. This object consists of two sets of white enamelled disks called **mires**, equally distant from the centre of a connecting bar. By increasing or diminishing the distance between the mires their separation may be made such that the double images are tangent to one another. From a scale which has been constructed by previous calculation, the curvature of the cornea or its refractive power in dioptres can be read off on the bar separating the mires.

The ophthalmoscope is a contrivance by which the observer reflects light into an eye, while he is in such position as to receive in his own eye the light which returns by reflexion and refraction from the observed eye. As Helmholtz invented the ophthalmometer, so to him are we indebted for the gift of the ophthalmoscope. Prior to this invention, in 1851, the question of seeing the fundus of the eye had attracted much attention. It was of course known that the eyes of some animals emit a reddish or greenish tint under certain circumstances, and

many absurd speculations were indulged in for the explanation of this phenomenon. Brücke made a thorough study of this subject, and in 1847 gave the true explanation.* Indeed, he came so near to the invention of the ophthalmoscope as to place in a flame an iron tube, through which he could see the fundus of the eye. It was also known prior to these discoveries that the fundus of the eye would become visible if the eye were immersed in water. The explanation of this is similar to that of the glow of a cat's eye.

The ophthalmoscope in its simplest form is a plane or concave mirror, having in its centre a small circular opening through which the observer receives the light returning from the observed eye. Reference to Fig. 52 shows why the fundus of an eye cannot be seen without a special contrivance. If light be reflected into the eye from a flame A along the axis AE, and along other axes near AE, a small image of the flame will be formed on the retina, and at the same time a small portion of the fundus adjacent to the image will be illuminated by irregular reflexion. Light from this portion of the fundus passes out of the eye. If the eye is emmetropic, the pencil from E, having as its axis EA, is after refraction changed to parallel rays; similarly, pencils from other points

* Mueller's "Archiv für Anat. und Phys.," 1845, S. 387; and 1847, SS. 225, 479.

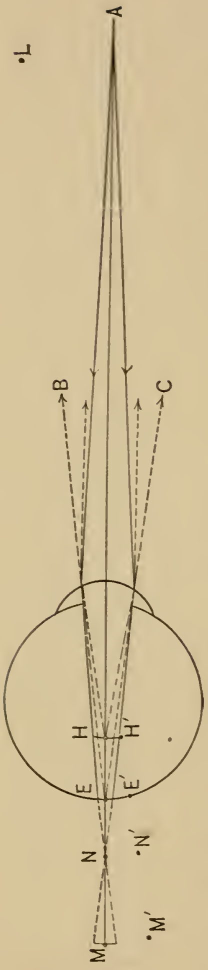

FIG. 52.

near E will have as axes lines lying very near EA, and these rays will also, after refraction, be parallel to their axes; thus it is evident that the light which returns from the examined eye cannot within a short distance from the eye deviate far from the axis EA. Hence we see that without a special contrivance the observer's head would necessarily cut off the light which illuminates the eye.

In the hyperopic eye pencils of light from any point on the fundus are divergent after leaving the eye. If the fundus be at H, then BHC will represent a pencil from H. If the rays diverge considerably it will be possible for an observer to place his head in position to receive some of the rays, and yet not obstruct those from the illuminating source. It is on this account that a cat's eye glows in the dark when the observer is not far from the path of the rays which enter the eye. Placing the eye under water has the same effect

upon pencils as hyperopia, for the refractive index of water is nearly the same as that of the aqueous, and, the external surface of the water being plane, we have in the immersed eye a high degree curvature hyperopia.

By referring to Fig. 52 we see that when the fundus is conjugate to the position from which the illumination proceeds, only a small portion of the fundus is illuminated, and that the more remote the fundus is from this conjugate position, the greater is the portion of fundus illuminated. This is the principle upon which is based the method of examination known as **skiascopy.** If the examination be conducted with a plane mirror so placed that light enters the eye in pencils diverging from a distance of one metre in front of the eye, and if the far point of the eye be also distant one metre, that is, if the eye have one dioptre of myopia, a very small part of the fundus will be illuminated. Hence, if the mirror be slightly tilted, the area of illumination will at once be thrown out of the line of vision of the observer. If the eye be hyperopic, then as the mirror is tilted the area of illumination will move in the same direction, but it will not pass out of view so rapidly, and we can observe the motion of the reflex and its attendant shadow as they move across the pupil. When the mirror is so tilted that the light which enters the eye appears to come from L, then a

straight line, drawn through the optical centre of the eye, connecting L, H', E', N', and M', replaces the axis AM; hence, no matter what may be the refractive condition of the eye, the area of illumination always moves in the direction of the tilting of the mirror. But in myopia such that the fundus is at M, beyond the conjugate N, there will be formed to the left of A an aerial image of the illuminated area, and this will evidently move in the opposite direction to the tilting of the mirror. If the observer is farther from the eye than this image, he must therefore see the reflex and shadow move in the opposite direction to the tilting.

By observing the movement of the shadow in different meridians of the eye, the test can also be used for the detection of astigmatism.

If a concave mirror be used so that light enters the eye diverging from a real image in front of the mirror, the movements of the shadow will be opposite to those which occur with the use of the plane mirror; the reason for this is apparent.

With such accuracy can the motion of the shadow be observed that this method, in skilful hands, surpasses all other objective means of examining the refractive condition of the eye.

Since light leaves the emmetropic eye in parallel rays and the hyperopic eye in divergent pencils, these rays may be brought to a focus on the retina

of the observer, thus forming an image of the illuminated part of the fundus under examination, so that in both hyperopia and emmetropia the details of the fundus can be seen, provided the observer is sufficiently near the examined eye to receive light from an appreciable area of the fundus. Moreover, since in hyperopia a larger part of the fundus is illuminated than in emmetropia, the details can be seen at a greater distance in hyperopic eyes.

The rays from hyperopic and emmetropic eyes, being either divergent or parallel, will never meet in a real aerial image; hence the image, as seen by the examiner, will always be erect. The examination of the erect image is called the **direct** method of ophthalmoscopy.

Light emerging from a myopic eye is convergent, and if the eye of an emmetropic observer be nearer the eye than the far point, a concave lens must be used to see clearly the details of the fundus. The image as thus seen will be erect; but at the far point of the eye, which we know is conjugate to the retina, an inverted aerial image will be formed, and from this, divergent pencils will enter an observer's eye situated beyond the image. A concave lens will be no longer required; on the other hand, exercise of the accommodation or a convex lens varying with the distance of the eye from the image is necessary. The examination of the inverted image is called the

indirect method of ophthalmoscopy. As thus described, this method would be practicable only in highly myopic eyes; but we may produce the aerial image in all cases by holding a strong convex lens in front of the eye to be examined. The stronger the lens the larger will be the field of view and the smaller the image; hence the strength of the converging lens may be varied to suit the purpose of the examiner.

Finally let us investigate the apparent size of the optic disk — the most conspicuous object revealed by the ophthalmoscope — as affected by the various refractive conditions of the eye. We shall first consider the examination by the direct method. When the eye under examination is emmetropic, no change in apparent size is produced by varying the distance between the observed and observer's eyes; for we see from Fig. 50 (in which $A'B'$ may represent the disk, the lens M may represent the observed, and HN the observer's eye respectively) that the visual angle HON under which the disk is seen does not vary with the distance between the two eyes.* The field of view becomes smaller as this distance increases, for more rays pass outside of the eye: but no change in size is produced. If the observed eye be hyperopic, rays after leaving it will be divergent, and this same divergence might be caused by placing a suitable

* See also Fig. 45, p. 170.

concave lens in contact with an emmetropic eye. A lens in this position is without the anterior focus of the observer's eye. We know that a concave lens placed without the anterior focus of an optical system diminishes the size of images formed by it, and that the minifying effect increases as the distance of the lens from the focus. Hence in hyperopia the disk appears smaller than in emmetropia, and its apparent size diminishes as the distance between the eyes increases. In myopia the pencils are convergent after leaving the eye, just as if a convex lens were placed before the eye; thus the disk in this case appears larger than in emmetropia, and the apparent size increases with the distance between the eyes.

In astigmatism the disk appears smaller than normal in the meridian of hyperopic and larger than normal in the meridian of myopic refraction. Hence supposing the disk to be circular in form, it appears oval in astigmatism with the long axis in the meridian of greatest refraction.

In the indirect method of examination the image as seen by the observer varies in size with the aerial image formed by the convergent lens; hence we must investigate the size of this image as affected by the refractive condition of the examined eye. In hyperopia the concave lens, which we suppose to be placed in contact with the cornea of a normal eye,

has a magnifying or minifying effect upon the aerial image according as the distance of the lens from the eye is less or greater than the focal length of the lens. Therefore in the indirect examination the disk appears larger than normal when the distance of the lens from the examined eye is less than the focal length of the lens; when this distance is equal to the focal length of the lens, the disk appears normal in size, and when the distance of the lens from the eye is greater than the focal length of the former, the disk appears smaller than normal.

In myopia we have the opposite conditions, that is, the disk appears diminished in size when the distance between the eye and lens is less than the focal length of the latter; and, increasing with the distance of the lens, the disk appears larger than normal when the distance of the lens from the eye is greater than the focal length of the lens.

Thus also in astigmatism the disk undergoes the opposite distortion to that which occurs in the direct examination, provided that the distance of the lens from the eye is less than the focal length of the lens, and when the lens is farther from the eye than this length the distortion is the same in both methods.

This distortion, first described by Knapp, was formerly used as a test for astigmatism. By placing a suitable lens before the eye the distortion can be made to disappear, and this lens represents the

amount of astigmatism present. While this test has given way to other more delicate ones, the phenomenon is, in high degrees of ametropia, so striking that it must attract the attention of every student of ophthalmoscopy.

APPENDIX I

For the convenience of those who may not be familiar with trigonometrical formulæ, we append the following synopsis:

In the figure, $ABCDE$ represents a quadrant, or 90 degrees of the circumference of a circle, of which AB, BC, CD, and DE are equal arcs. The angles AOB, BOC, COD, and DOE are also equal.

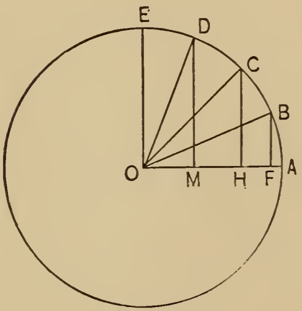

FIG. 53.

If BF be drawn perpendicular to OA, then $\frac{BF}{OB}$ is called the **sine** of the angle AOB; similarly, $\frac{CH}{OC}$ is the sine of COH. $\frac{OF}{OB}$ is called the **cosine** of AOB, $\frac{BF}{OF}$ is the **tangent**, and $\frac{OF}{BF}$ the **cotangent** of this angle.

The angle AOC is twice the angle AOB, but it is readily seen that the perpendicular CH is less than twice the perpendicular BF; hence the sine of twice an angle is less than twice the sine of the angle. Similarly

the sine of AOD is less than three times the sine of AOB; moreover it is evident that the increase in the sine caused by adding the angle COD is less than that caused by adding the equal angle BOC. As the angle approaches 90 degrees, an increase in this angle will produce much less increase in the length of the perpendicular DM than the same increase of a small angle. The opposite to this is true of the cosine of an angle; as the angle increases the cosine diminishes, and at an increasing rate as the angle approaches 90 degrees.

The sine of an angle increases, as we see, from zero to unity as the angle increases from zero to 90 degrees, and the cosine diminishes from unity to zero as the angle increases from zero to 90 degrees. The value of the sine, cosine, or tangent of any angle can be found from tables which have been constructed by calculation.

All the trigonometrical formulæ used in the preceding pages are easily deduced from the foregoing. They are as follows:

$$\tan a = \frac{\sin a}{\cos a}; \ \cot a = \frac{\cos a}{\sin a};$$

$$\sin(a \pm b) = \sin a \cdot \cos b \pm \cos a \sin b; \ \sin 2a = 2 \sin a \cdot \cos a;$$
$$\sin(180 - a) = \sin a; \ \sin^2 a + \cos^2 a = 1.$$

Also in any triangle ABC, $\frac{\sin A}{\sin B} = \frac{BC}{AC}$, BC being the side opposite to the angle A, and AC being opposite to B.

APPENDIX II

The following demonstration is based upon that given in Heath's "Geometrical Optics": *

In the figure, the normal, MN, to the refracting surface lies in the plane of the paper represented by $ABCD$, and

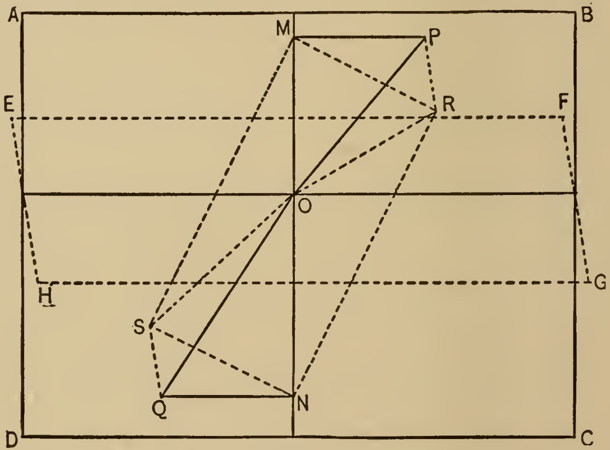

Fig. 54.

the plane $EFGH$ is perpendicular to the plane of the paper. For the sake of clearness all lines which lie in

* "Geometrical Optics," Heath, 2d ed., p. 21.

the plane $ABCD$ are drawn as continuous, and all not in this plane are drawn as interrupted lines.

Let RO represent an incident ray which lies in front of the plane $ABCD$, then OS, the refracted ray, will lie behind this plane. If the first medium be air, whose index is 1, and if the index of the second medium be n, then we shall have the equation

$$\sin MOR = n \cdot \sin SON.$$

Since the incident and refracted rays must lie in the plane $MRNS$, the deviation produced by the refraction is in this plane, but we have learned that a deviation in any plane may be resolved into two deviations at right angles to each other; and therefore a deviation in the plane $MRNS$ may be considered as the resultant of two deviations lying respectively in the plane $ABCD$ and in $PQRS$ at right angles to $ABCD$. To find these resultant deviations, lay off the distances RO and OS such that $OS = n \cdot RO$. From R, which lies in front of the plane $ABCD$, draw RM perpendicular to MN, and RP perpendicular to $ABCD$; from S, which lies behind the plane $ABCD$, draw in like manner SN and SQ. Then as RMO is a right angle, we have $MR = RO \cdot \sin MOR$, and from the triangle SON we have $SN = OS \cdot \sin SON$, or, since $OS = n \cdot RO$, $SN = RO \cdot n \cdot \sin SON$; but $\sin MOR = n \cdot \sin SON$, hence $MR = SN$. MR is also parallel to SN, since both lie in the same plane and are perpendicular to MN. We see also that PR is parallel to SQ and PM to QN; thus the triangles PMR and NQS are equal, and $PR = QS$ and $PM = QN$.

Let the angle POR, which the incident ray makes with the plane $ABCD$, be denoted by i_1, and let the angle QOS, which the refracted ray makes with this plane, be denoted by r_1; then $PR = RO \cdot \sin i_1$, and $QS = OS \cdot \sin r_1$, or, since $PR = QS$ and $OS = n \cdot RO$, we have

$$\sin i_1 = n \cdot \sin r_1. \tag{1}$$

From this we see that there is the same relation between i_1 and r_1 as between the angles of incidence and refraction, MOR and SON.

If the angle POM be denoted by i_2, and QON by r_2, we shall have $PM = PO \cdot \sin i_2 = RO \cdot \cos i_1 \cdot \sin i_2$, and $QN = QO \cdot \sin r_2 = OS \cdot \cos r_1 \sin r_2 = n \cdot RO \cdot \cos r_1 \sin r_2$.

From this, since $PM = QN$, we have

$$\cos i_1 \sin i_2 = n \cdot \cos r_1 \sin r_2,$$

or
$$\sin i_2 = n \cdot \frac{\cos r_1}{\cos i_1} \cdot \sin r_2. \tag{2}$$

Thus we see that i_2 and r_2 are connected by the law of refraction, the index of the first medium being 1 and that of the second being $n \cdot \frac{\cos r_1}{\cos i_1}$. From equations (1) and (2) we can find the deviation of the ray ROS in the planes $PRQS$ and $MPNQ$, or $ABCD$.

When n is greater than unity, as in glass, it follows from (1) that i_1 is greater than r_1, and since the greater the angle the smaller is the cosine, $\frac{\cos r_1}{\cos i_1}$ must be greater than unity; hence $n \cdot \frac{\cos r_1}{\cos i_1}$ is greater than n. In other

words, the deviation of the ray in the direction of the plane *ABCD* is the same as would be produced in a ray, *PO*, upon entering a medium of greater index than that of the medium which we are considering. As we know that the deviation increases with the index, it follows that the ray *RO* undergoes a greater deviation in the plane *ABCD* than would a ray in the line *PO*, which is the projection of *RO* on this plane.

We can now apply these deductions to refraction through a prism. The plane *ABCD* represents a principal plane of the prism; the ray *RO*, not in this plane, will, at the first face of the prism, undergo deviation in the plane *MRNS*, and this deviation is equivalent to a certain deviation in the principal plane superposed upon a deviation in the plane *PRQS*, which is perpendicular to the principal plane. The plane *PRQS*, being perpendicular to the principal plane, cuts the two faces of the prism in two parallel lines. From equation (1) we have seen that the angles *POR* and *QOS*, whose difference expresses the deviation in the plane *PRQS*, are connected by the law of refraction; and, since this plane cuts the faces of the prism in parallel lines, it is clear that the deviation in this plane at the first face must be neutralized by that at the second face. Thus we see that in any position of the ray no deviation is produced in the direction of the edge of the prism.*

Let us now investigate the deviation by the prism in

* We have already assumed this to be true for the cylindrical lens, which we may consider as composed of an infinite number of prisms whose edges are all parallel to the axis of the lens.

its principal plane. The edge of the prism would be represented by a line perpendicular to $ABCD$; such a line would be parallel to PR, and consequently R and P are equally distant from the edge. At the first face the deviation in the principal plane is greater for the ray RO than for a ray, PO, equally distant from the edge of the prism. We have seen that the deviation in a principal plane which the ray RO undergoes at the first face is the same as would be produced in the ray PO upon entering a medium whose index is $n \cdot \frac{\cos r_1}{\cos i_1}$. We have seen also that in its passage through the prism there is no deviation of the ray in the plane $PRQS$; hence, if we trace the ray backward, we shall have at the second face the same angles, POR and QOS, that we have at the first face. At the second face also, then, the oblique ray is deviated in the principal plane to the same extent as the corresponding ray in this plane would be deviated by a prism whose index is $n \cdot \frac{\cos r_1}{\cos i_1}$. Since the total deviation produced by a prism increases as the index increases, it is proved that the deviation in the principal plane is greater for rays not in this plane than for those which lie in it.

When POR (i_1) increases, the index of the hypothetical prism increases at a continually increasing rate.

INDEX

DEFECTIVE EYESIGHT:

The Principles of Its Relief by Glasses.

BY

D. B. ST. JOHN ROOSA, M.D., LL.D.,

Professor Emeritus of Diseases of the Eye and Ear; Post-Graduate Medical School and Hospital; Surgeon to the Manhattan Eye and Ear Hospital, etc., etc.; author of "A Clinical Manual of Diseases of the Eye"; "Ophthalmic and Otic Memoranda"; "A Practical Treatise on the Diseases of the Ear"; "The Old Hospital and Other Papers"; "A Vest-Pocket Medical Lexicon."

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